Séminaire Lotharingien de Combinatoire **89B** (2023)
Article #36, 12 pp.

Proceedings of the 35th Conference on Formal Power Series and Algebraic Combinatorics (Davis)

Hilbert–Poincaré series of matroid Chow rings and intersection cohomology

Luis Ferroni*[1], Jacob P. Matherne†[2,3], Matthew Stevens‡[2], and Lorenzo Vecchi§[4]

[1] *Department of Mathematics, KTH Royal Institute of Technology, Sweden*
[2] *Mathematical Institute, University of Bonn, Germany*
[3] *Max Planck Institute for Mathematics, Bonn, Germany*
[4] *Dipartimento di Matematica, Università di Bologna, Italy*

Abstract. We study the Hilbert–Poincaré series of three algebraic objects arising in the Chow-theoretic and Kazhdan–Lusztig framework of matroids. These are, respectively, the Hilbert–Poincaré series of the Chow ring, the augmented Chow ring, and the intersection cohomology module. We develop and highlight an explicit parallelism between the Kazhdan–Lusztig polynomial of a matroid and the Hilbert–Poincaré series of its Chow ring that extends naturally to the Hilbert–Poincaré series of both the intersection cohomology module and the augmented Chow ring.

Keywords: Matroids, Chow rings, Kazhdan–Lusztig polynomials, Hilbert–Poincaré series, Binomial Eulerian polynomials, Real-rootedness, γ-positivity

1 Introduction

Starting from a loopless matroid M, one can construct various algebraic objects having good properties that in turn can be used to answer purely combinatorial questions. Three notable examples of such objects are the **Chow ring** $\underline{\mathrm{CH}}(\mathsf{M})$, the **augmented Chow ring** $\mathrm{CH}(\mathsf{M})$, and the **intersection cohomology module** $\mathrm{IH}(\mathsf{M})$. These three algebraic structures possess a number of remarkable features and play an instrumental role in the proofs of the log-concavity of the Whitney numbers of the first kind [1], and the top-heaviness of the Whitney numbers of the second kind of the lattice of flats $\mathscr{L}(\mathsf{M})$ [7]. Our main objects of study in this article are the coefficients of the respective Hilbert–Poincaré series of each of these three structures.

*ferroni@kth.se. LF is supported by the Swedish Research Council grant 2018-03968.

†jacobm@math.uni-bonn.de. JM is supported by the Deutsche Forschungsgemeinschaft (DFG) under Germany's Excellence Strategy - GZ 2047/1, Projekt-ID 390685813.

‡s6mastev@uni-bonn.de.

§lorenzo.vecchi6@unibo.it.

The intersection cohomology module $\mathrm{IH}(\mathsf{M})$ is particularly relevant in the Kazhdan–Lusztig theory of matroids. Its Hilbert–Poincaré series is known in the literature as the **Z-polynomial** of the matroid M and is denoted by $Z_{\mathsf{M}}(x)$. The history is as follows. First, in [23], Proudfoot, Xu, and Young introduced the Z-polynomial using a purely combinatorial language (i.e., via the displayed equation in Theorem 1 (iii) below), without making reference to any notion of intersection cohomology for arbitrary matroids (at that point). Later, in [7], Braden, Huh, Matherne, Proudfoot, and Wang introduced the intersection cohomology module $\mathrm{IH}(\mathsf{M})$ of a matroid M and showed that the combinatorially-defined $Z_{\mathsf{M}}(x)$ is equal to the Hilbert–Poincaré series of $\mathrm{IH}(\mathsf{M})$.

In fact, the Z-polynomial is strongly related to the **Kazhdan–Lusztig polynomial** of M, denoted $P_{\mathsf{M}}(x)$ and first studied by Elias, Proudfoot, and Wakefield in [11]. The following can be used as a simultaneous recursive definition for both the Kazhdan–Lusztig and Z-polynomials of matroids.

Theorem 1 ([23, 8])**.** *There is a unique way to assign to each loopless matroid* M *a polynomial* $P_{\mathsf{M}}(x) \in \mathbb{Z}[x]$ *such that the following properties hold:*

(i) If $\mathrm{rk}(\mathsf{M}) = 0$, *then* $P_{\mathsf{M}}(x) = 1$.

(ii) If $\mathrm{rk}(\mathsf{M}) > 0$, *then* $\deg P_{\mathsf{M}}(x) < \frac{1}{2}\,\mathrm{rk}(\mathsf{M})$.

(iii) For every matroid M, *the polynomial*

$$Z_{\mathsf{M}}(x) := \sum_{F \in \mathscr{L}(\mathsf{M})} x^{\mathrm{rk}(F)}\, P_{\mathsf{M}/F}(x)$$

is palindromic.[1]

The original definition of the Kazhdan–Lusztig polynomials of Elias, Proudfoot, and Wakefield is given by the following result, which provides a recursion that defines them uniquely in terms of characteristic polynomials of restrictions $\mathsf{M}|_F$ and Kazhdan–Lusztig polynomials of contractions M/F for flats $F \in \mathscr{L}(\mathsf{M})$. This does not make any reference to the Z-polynomial.

Theorem 2 ([11])**.** *There is a unique way to assign to each loopless matroid* M *a polynomial* $P_{\mathsf{M}}(x) \in \mathbb{Z}[x]$ *such that the following conditions hold:*

(i) If $\mathrm{rk}(\mathsf{M}) = 0$, *then* $P_{\mathsf{M}}(x) = 1$.

(ii) If $\mathrm{rk}(\mathsf{M}) > 0$, *then* $\deg P_{\mathsf{M}}(x) < \frac{1}{2}\,\mathrm{rk}(\mathsf{M})$.

[1] In this context, we say that $p(x)$ is palindromic if $p(x) = x^d p(x^{-1})$ where $d = \deg p(x)$. For example, the polynomial $q(x) = x^2 + x$ is *not* palindromic, even though it satisfies $x^3 q(x^{-1}) = q(x)$.

(iii) For every matroid M*, the following recursion holds:*

$$x^{\operatorname{rk}(\mathsf{M})} P_{\mathsf{M}}(x^{-1}) = \sum_{F \in \mathscr{L}(\mathsf{M})} \chi_{\mathsf{M}|_F}(x)\, P_{\mathsf{M}/F}(x).$$

These results provide compact and purely combinatorial definitions of $P_{\mathsf{M}}(x)$ and $Z_{\mathsf{M}}(x)$. However, many properties of these families of polynomials are not easy to deduce from such statements. For instance, a non-obvious property that these polynomials possess is the nonnegativity of their coefficients. In fact, another important result of [7] precisely establishes this nonnegativity by proving that the coefficients of these polynomials are given by certain graded dimensions; more precisely, $Z_{\mathsf{M}}(x) = \operatorname{Hilb}(\operatorname{IH}(\mathsf{M}), x)$ and $P_{\mathsf{M}}(x) = \operatorname{Hilb}(\operatorname{IH}(\mathsf{M})_{\varnothing}, x)$ (see [7, Theorem 1.9]).

We now turn history on its head. Recall that in the Kazhdan–Lusztig setting, the combinatorially-defined Kazhdan–Lusztig and Z-polynomials came first, and their descriptions as Hilbert–Poincaré series of $\operatorname{IH}(\mathsf{M})_{\varnothing}$ and $\operatorname{IH}(\mathsf{M})$ came later. For the Chow-theoretic setting, the Chow ring $\underline{\operatorname{CH}}(\mathsf{M})$ and augmented Chow ring $\operatorname{CH}(\mathsf{M})$ were first introduced in [12] and [6], respectively. In what follows, we give an "intrinsic" combinatorial definition of their Hilbert–Poincaré series by mirroring Theorem 1 and Theorem 2. (Theorem 5 below asserts that these polynomials are indeed the correct Hilbert–Poincaré series.) These combinatorial definitions can be seen as the point of departure of our study.

Theorem 3. *There is a unique way to assign to each loopless matroid* M *a palindromic polynomial* $\underline{\mathrm{H}}_{\mathsf{M}}(x) \in \mathbb{Z}[x]$ *such that the following properties hold:*

(i) If $\operatorname{rk}(\mathsf{M}) = 0$*, then* $\underline{\mathrm{H}}_{\mathsf{M}}(x) = 1$.

(ii) If $\operatorname{rk}(\mathsf{M}) > 0$*, then* $\deg \underline{\mathrm{H}}_{\mathsf{M}}(x) = \operatorname{rk}(\mathsf{M}) - 1$.

(iii) For every matroid M*, the polynomial*

$$\mathrm{H}_{\mathsf{M}}(x) := \sum_{F \in \mathscr{L}(\mathsf{M})} x^{\operatorname{rk}(F)}\, \underline{\mathrm{H}}_{\mathsf{M}/F}(x)$$

is palindromic.

Similar to the case of the Kazhdan–Lusztig polynomials and the Z-polynomials, a non-trivial property that one is able to observe is that the polynomials $\underline{\mathrm{H}}_{\mathsf{M}}(x)$, and therefore also the polynomials $\mathrm{H}_{\mathsf{M}}(x)$, appear to have nonnegative coefficients. On the other hand, given the resemblance between Theorem 3 and Theorem 1, it is reasonable to ask for a counterpart for Theorem 2 in this alternative setting.

Theorem 4. *There is a unique way to assign to each loopless matroid* M *a polynomial* $\underline{\mathrm{H}}_{\mathsf{M}}(x) \in \mathbb{Z}[x]$ *such that the following conditions hold:*

(i) *If* $\operatorname{rk}(\mathsf{M}) = 0$, *then* $\underline{\mathrm{H}}_{\mathsf{M}}(x) = 1$.

(ii) *For every matroid* M, *the following recursion holds:*

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = \sum_{\substack{F \in \mathcal{L}(\mathsf{M}) \\ F \neq \varnothing}} \overline{\chi}_{\mathsf{M}|_F}(x)\, \underline{\mathrm{H}}_{\mathsf{M}/F}(x).$$

As it is stated, the proof of this fact is an easy induction. What is much less evident is that the polynomials $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ arising from this result coincide with those arising from Theorem 3. In particular, it is already a non-trivial and interesting conclusion that the polynomials as defined in Theorem 4 are palindromic, as this is not at all hinted by the recursion. A slight difference with respect to Theorem 2 that deserves a comment is that the polynomial $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ is defined recursively via a convolution of itself with the *reduced* characteristic polynomial. Again, the nonnegativity of the coefficients of $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ is not easy to deduce from this recursion and, moreover, the fact that the reduced characteristic polynomial has coefficients that alternate in sign introduces an additional issue.

The polynomials $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ are less manageable than $P_{\mathsf{M}}(x)$ and $Z_{\mathsf{M}}(x)$ from a computational point of view. For example neither $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ nor $\mathrm{H}_{\mathsf{M}}(x)$ behave well under direct sums of matroids; one possible explanation for this is that the reduced characteristic polynomial of a direct sum is not the product of the reduced characteristic polynomials of the summands and, in particular, the argument that Elias, Proudfoot, and Wakefield used in [11, Proposition 2.7] to prove that $P_{\mathsf{M}_1 \oplus \mathsf{M}_2}(x) = P_{\mathsf{M}_1}(x) P_{\mathsf{M}_2}(x)$ does not hold.

The key step towards deducing that the coefficients of these polynomials are always nonnegative integers is given by the following connection with the Chow ring and the augmented Chow ring.

Theorem 5. *Let* M *be a loopless matroid. The polynomial* $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ *is the Hilbert–Poincaré series of the Chow ring* $\underline{\mathrm{CH}}(\mathsf{M})$ *of* M. *The polynomial* $\mathrm{H}_{\mathsf{M}}(x)$ *is the Hilbert–Poincaré series of the augmented Chow ring* $\mathrm{CH}(\mathsf{M})$. *In particular, both of them have nonnegative coefficients.*

The proof relies essentially on a construction of Feichtner and Yuzvinsky [12] of a certain Gröbner basis for the Chow ring of atomic lattices with respect to arbitrary building sets. The strategy is to start with a raw expression for the Hilbert–Poincaré series of both the Chow ring and the augmented Chow ring and prove that they satisfy the recursions of both Theorem 3 and Theorem 4; this is very much in resemblance to the proof of the positivity of the coefficients of the Kazhdan–Lusztig and the Z-polynomials in [7, Theorem 1.2].

Remark 6. *The reduced characteristic polynomial is a Tutte–Grothendieck invariant of the matroid, i.e., the Tutte polynomial of the matroid determines it. A natural question that one could ask is whether the Hilbert–Poincaré series of the Chow ring has the same property. As can be* *seen from Theorem 4, the map* $\mathsf{M} \mapsto \underline{\mathrm{H}}_{\mathsf{M}}(x)$ *is essentially the inverse of the map* $\mathsf{M} \mapsto \overline{\chi}_{\mathsf{M}}(x)$ *in the incidence algebra of the lattice of flats. In spite of that, we can find two matroids* M_1 *and* M_2 *of rank* 4 *on* 7 *elements having the same Tutte polynomial but whose Chow rings have different Hilbert–Poincaré series. Precisely, consider the matroids* M_1^* *and* M_2^* *depicted in Figure 1 (the reason for depicting the duals instead of the original matroids is that they have rank* 3*):*

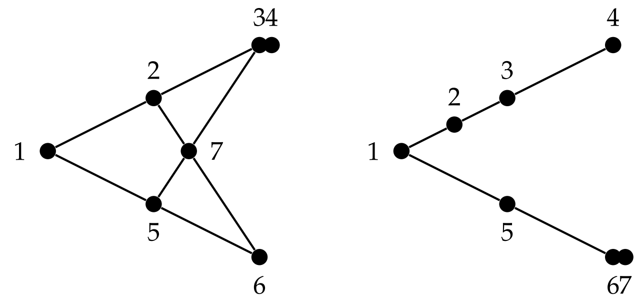

Figure 1: The duals of the matroids M_1 and M_2

The matroids M_1 *and* M_2 *have the same Tutte polynomial:*

$$T_{\mathsf{M}_1}(x,y) = T_{\mathsf{M}_2}(x,y) = x^4 + 3x^3 + 2x^2y + xy^2 + y^3 + 4x^2 + 5xy + 3y^2 + 2x + 2y.$$

However, we have

$$\underline{\mathrm{H}}_{\mathsf{M}_1}(x) = x^3 + 30x^2 + 30x + 1,$$
$$\underline{\mathrm{H}}_{\mathsf{M}_2}(x) = x^3 + 31x^2 + 31x + 1.$$

Moreover, the same example shows that the Hilbert–Poincaré series of the augmented Chow ring is not an evaluation of the Tutte polynomial because

$$\mathrm{H}_{\mathsf{M}_1}(x) = x^4 + 37x^3 + 98x^2 + 37x + 1,$$
$$\mathrm{H}_{\mathsf{M}_2}(x) = x^4 + 38x^3 + 102x^2 + 38x + 1.$$

We also mention explicitly the fact that neither $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ *nor* $\mathrm{H}_{\mathsf{M}}(x)$ *determines the other.*

2 Real-rootedness and gamma-positivity

As mentioned earlier, one can view Theorem 3 and Theorem 4 as “intrinsic” combinatorial definitions of the Hilbert–Poincaré series of the rings $\underline{\mathrm{CH}}(\mathsf{M})$ and $\mathrm{CH}(\mathsf{M})$, avoiding their explicit construction.

Given the similarity that we have with the counterpart definitions in the Kazhdan–Lusztig framework, some statements (many of which are still conjectural) regarding the Kazhdan–Lusztig polynomial and the Z-polynomial of matroids now admit natural analogues for the Chow ring and the augmented Chow ring.

Two outstanding conjectures posed in [16] and [23] assert the **real-rootedness** of both $P_{\mathsf{M}}(x)$ and $Z_{\mathsf{M}}(x)$. A well known fact (see, e.g., [9]) is that the real-rootedness of a polynomial with positive coefficients implies that the coefficients form an **ultra log-concave sequence** and, in the case the polynomial is palindromic, that it is **γ-positive**. Each of these two properties, i.e., ultra log-concavity or γ-positivity, is also known to be stronger than the **unimodality** of the coefficients of the polynomial.

The similarity between the defining recursions of the polynomials arising in Theorem 1 and Theorem 3, and a significant amount of experimentation, make it plausible to postulate that these desirable properties of the coefficients might hold for $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ as well. Indeed, these conjectural properties had already been observed in the literature.

Conjecture 7. *For every matroid* M*, the following polynomials have only real roots.*

- *([14, Conjecture 10.19]) The polynomial* $\underline{\mathrm{H}}_{\mathsf{M}}(x)$*, i.e., the Hilbert–Poincaré series of the Chow ring* $\underline{\mathrm{CH}}(\mathsf{M})$*.*
- *([25, Conjecture 4.3.3]) The polynomial* $\mathrm{H}_{\mathsf{M}}(x)$*, i.e., the Hilbert–Poincaré series of the augmented Chow ring* $\mathrm{CH}(\mathsf{M})$*.*
- *([23, Conjecture 5.1]) The polynomial* $Z_{\mathsf{M}}(x)$*, i.e., the Hilbert–Poincaré series of the intersection cohomology module* $\mathrm{IH}(\mathsf{M})$*.*
- *([16, Conjecture 3.2]) The polynomial* $P_{\mathsf{M}}(x)$*, i.e., the Hilbert–Poincaré series of the stalk* $\mathrm{IH}(\mathsf{M})_{\varnothing}$ *at the empty flat of* $\mathrm{IH}(\mathsf{M})$[2].

The main result of Adiprasito, Huh, and Katz [1, Theorem 1.4] asserts the validity of the **Kähler package** for the Chow rings of matroids. In particular, **Poincaré duality** and the **hard Lefschetz theorem** are valid, and they imply respectively the palindromicity and the unimodality of the coefficients of the Hilbert–Poincaré series of the Chow ring. On the other hand, the Kähler package was proved for the augmented Chow ring by Braden, Huh, Matherne, Proudfoot, and Wang in [6, Theorem 1.6] and for the intersection cohomology module in [7, Theorem 1.6], so one can conclude analogous statements for the Hilbert–Poincaré series of $\mathrm{CH}(\mathsf{M})$ and $\mathrm{IH}(\mathsf{M})$. Less is known regarding the Hilbert–Poincaré series of $\mathrm{IH}(\mathsf{M})_{\varnothing}$, i.e., the Kazhdan–Lusztig polynomial $P_{\mathsf{M}}(x)$. Although its coefficients are nonnegative, it is not known whether they are always unimodal.

[2] We note that $\mathrm{CH}(\mathsf{M})$ and $\mathrm{IH}(\mathsf{M})$ are both modules over the graded Möbius algebra $\mathrm{H}(\mathsf{M})$. We make this point only to be able to define $\mathrm{IH}(\mathsf{M})_{\varnothing}$, which depends on the $\mathrm{H}(\mathsf{M})$-module structure. As the purpose of this paper is to study various Hilbert–Poincaré series, we only need to view $\underline{\mathrm{H}}(\mathsf{M})$, $\mathrm{H}(\mathsf{M})$, $\underline{\mathrm{CH}}(\mathsf{M})$, $\mathrm{CH}(\mathsf{M})$, $\mathrm{IH}(\mathsf{M})$, and $\mathrm{IH}(\mathsf{M})_{\varnothing}$ as graded vector spaces.

Another of our contributions will be to provide a proof of the γ-positivity for three of the families of polynomials above (the palindromic ones, those for which this statement is meaningful).

Theorem 8. *For every matroid* M*, the polynomials* $\underline{\mathrm{H}}_{\mathsf{M}}(x)$*,* $\mathrm{H}_{\mathsf{M}}(x)$*, and* $Z_{\mathsf{M}}(x)$ *are* γ*-positive.*

This result gives further evidence to part of Conjecture 7, and resolves a problem posed in [13, Conjecture 5.6] by Ferroni, Nasr, and Vecchi that was known to hold only in certain cases. The proofs of the γ-positivity of each of these families are surprisingly simple, and depend only on a recursion witnessed by the validity of the semi-small decompositions of [6] for the Chow ring and the augmented Chow ring. For the intersection cohomology, a recursion found by Braden and Vysogorets [8] does the job.

Remark 9. *Two natural questions that are reasonable to formulate are whether the Kazhdan–Lusztig polynomial* $P_{\mathsf{M}}(x)$ *is a "non-symmetric* γ*-positive" polynomial, in the sense of [3, Section 5.1], and whether the equivariant Z-polynomial (see [23, Section 6]) is "equivariant* γ*-positive", in the sense of [3, Section 5.2]. The answer to both of these questions is negative, and it is possible to provide explicit counterexamples.*

3 Uniform matroids, binomial Eulerian polynomials, and matroid polytopes

The case of uniform matroids is of particular interest both in the Kazhdan–Lusztig and Chow-theoretic frameworks.

For the Z-polynomial and the Kazhdan–Lusztig polynomial of uniform matroids, some explicit formulas are known (see e.g. [15]); we point out that the real-rootedness of $P_{\mathsf{M}}(x)$ and $Z_{\mathsf{M}}(x)$ for $\mathsf{M} \cong \mathsf{U}_{k,n}$ are still open problems.

For the Hilbert–Poincaré series of both the Chow ring and the augmented Chow ring, the situation is also complicated. The polynomials $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ for arbitrary uniform matroids were addressed by Hameister, Rao, and Simpson in [18]; they give a nice description of $\underline{\mathrm{H}}_{\mathsf{U}_{k,n}}(x)$ in terms of statistics of permutations, but this description turns out to be quite intricate from a computational point of view.

For Boolean matroids, the polynomial $\mathrm{H}_{\mathsf{M}}(x)$ has been studied in detail recently in [22, 4, 24, 17, 19, 10]. To be precise, for a Boolean matroid $\mathsf{M} \cong \mathsf{U}_{n,n}$ one has that $\underline{\mathrm{H}}_{\mathsf{M}}(x) = A_n(x)$, the n-th *Eulerian polynomial* (named this way in [24]), whereas $\mathrm{H}_{\mathsf{M}}(x) = \widetilde{A}_n(x)$ is the n-th *binomial Eulerian polynomial*; notice that the recursion in Theorem 3 (iii) asserts the identity

$$\widetilde{A}_n(x) = \sum_{j=0}^{n} \binom{n}{j} x^j \, A_{n-j}(x).$$

One can see the Hilbert–Poincaré series of Chow rings and augmented Chow rings as vast generalizations of these polynomials, and within this broader framework one can derive some interesting identities between them.

The polynomials $P_{\mathsf{M}}(x)$, $Z_{\mathsf{M}}(x)$, $\underline{\mathrm{H}}_{\mathsf{M}}(x)$, and $\mathrm{H}_{\mathsf{M}}(x)$ are defined recursively. In particular, computing them for fairly small matroids is already a challenging computational task. Towards simplifying their computation, a further contribution we make is extending a result of Ardila and Sanchez [2, Theorem 8.9] by proving the following result for $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$.

Theorem 10. *The maps $\mathsf{M} \mapsto \underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathsf{M} \mapsto \mathrm{H}_{\mathsf{M}}(x)$ are valuative under matroid polytope subdivisions.*

In other words, one can compute the Hilbert–Poincaré series of the augmented Chow ring of a matroid M by subdividing its base polytope $\mathcal{P}(\mathsf{M})$ into smaller matroid polytopes and computing it for those smaller pieces. As a consequence, by using the notion of relaxations of stressed subsets of Ferroni and Schröter [14], one proves that the Hilbert–Poincaré series of the Chow ring and augmented Chow ring behave well under relaxations of stressed subsets (in particular, stressed hyperplanes). This in turn yields fast ways of computing $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ for certain conjecturally "predominant" classes of matroids.[3] In particular, for the class of paving matroids, we obtain the following expressions in analogy to [13, Theorem 1.4].

Theorem 11. *Let M be a paving matroid of rank k and cardinality n. Then,*

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) = \underline{\mathrm{H}}_{\mathsf{U}_{k,n}}(x) - \sum_{h\geq k} \lambda_h \left(\underline{\mathrm{H}}_{\mathsf{U}_{k,h+1}}(x) - \underline{\mathrm{H}}_{\mathsf{U}_{k-1,h}\oplus\mathsf{U}_{1,1}}(x)\right),$$
$$\mathrm{H}_{\mathsf{M}}(x) = \mathrm{H}_{\mathsf{U}_{k,n}}(x) - \sum_{h\geq k} \lambda_h \left(\mathrm{H}_{\mathsf{U}_{k,h+1}}(x) - \mathrm{H}_{\mathsf{U}_{k-1,h}\oplus\mathsf{U}_{1,1}}(x)\right),$$

where λ_h denotes the number of stressed hyperplanes of size h in M.

In order to make these formulas useful in practice, we prove a conjecture of Hameister, Rao, and Simpson [18, Conjecture 6.2]; this and its analogue in the augmented case yield an explicit formula for $\underline{\mathrm{H}}_{\mathsf{U}_{k,n}}(x)$ and $\mathrm{H}_{\mathsf{U}_{k,n}}(x)$ for arbitrary uniform matroids. We state an equivalent reformulation now, which also highlights the connection with (binomial) Eulerian polynomials.

[3] In the terminology of Mayhew et al. [21], we mean that asymptotically almost every matroid lies in the alleged class.

Theorem 12. *The Hilbert–Poincaré series of the Chow ring and the augmented Chow ring of a uniform matroid of rank k and cardinality n are given by*

$$\underline{\mathrm{H}}_{\mathsf{U}_{k,n}}(x) = \sum_{j=0}^{k-1} \binom{n}{j} A_j(x)\, \overline{\chi}_{\mathsf{U}_{k-j,n-j}}(x),$$
$$\mathrm{H}_{\mathsf{U}_{k,n}}(x) = \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{n}{j} \binom{n-1-j}{k-1-j} \widetilde{A}_j(x)\, \frac{1-x^{k-j+1}}{1-x}.$$

These formulas can be plugged into the equations in Theorem 11 to provide a very fast way of computing the Hilbert–Poincaré series of Chow rings and augmented Chow rings of all paving matroids. In particular, we can test with a computer the validity of Conjecture 7 for sparse paving matroids with up to 40 elements.[4]

Proposition 13. *Let* M *be a sparse paving matroid on a ground set with at most* 40 *elements. Then both* $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ *and* $\mathrm{H}_{\mathsf{M}}(x)$ *are real-rooted polynomials.*

4 Monotonicity on coefficients

An unpublished conjecture posed by Gedeon asserts that the maximum of the coefficients of the Kazhdan–Lusztig and Z-polynomial of a matroid of rank k and size n is attained when the matroid is isomorphic to $\mathsf{U}_{k,n}$; this conjecture first appeared in print in [20, Conjecture 1.2]. It is known to hold for all paving matroids [13, Theorem 1.5]. Here we prove that its counterpart for the polynomials $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ indeed holds for all matroids.

Theorem 14. *Let* M *be a loopless matroid of rank k on a ground set of size n. The following coefficient-wise inequalities hold:*

$$\underline{\mathrm{H}}_{\mathsf{M}}(x) \preceq \underline{\mathrm{H}}_{\mathsf{U}_{k,n}}(x),$$
$$\mathrm{H}_{\mathsf{M}}(x) \preceq \mathrm{H}_{\mathsf{U}_{k,n}}(x).$$

In other words, uniform matroids maximize coefficient-wisely the coefficients of the Hilbert–Poincaré series of Chow rings and augmented Chow rings among all matroids with fixed rank and size.

The key gadget to prove this conjecture is given by the construction of an injective map from the barycentric subdivision of the lattice of flats of each matroid of rank k

[4] We observe that, using a rough estimation [5, Equation (10)], there are approximately $2^{2^{40}/60} \approx 10^{5000000000}$ sparse paving matroids on a ground set of cardinality 40.

Table 1: Examples of $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$ for some uniform matroids M

(a) Examples of $\underline{\mathrm{H}}_{\mathsf{U}_{k,k+1}}(x)$

$k =$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
x		1	7	21	51	113	239
x^2			1	21	161	813	3361
x^3				1	51	813	7631
x^4					1	113	3361
x^5						1	239
x^6							1

(b) Examples of $\mathrm{H}_{\mathsf{U}_{k,k+1}}(x)$

$k =$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
x^1	1	4	11	26	57	120	247
x^2		1	11	66	302	1191	4293
x^3			1	26	302	2416	15619
x^4				1	57	1191	15619
x^5					1	120	4293
x^6						1	247
x^7							1

(c) Examples of $\underline{\mathrm{H}}_{\mathsf{U}_{k,k+2}}(x)$

$k =$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
x^1		1	11	36	92	211	457
x^2			1	36	337	1877	8269
x^3				1	92	1877	20155
x^4					1	211	8269
x^5						1	457
x^6							1

(d) Examples of $\mathrm{H}_{\mathsf{U}_{k,k+2}}(x)$

$k =$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
x^1	1	5	16	42	99	219	466
x^2		1	16	117	610	2641	10204
x^3			1	42	610	5637	40444
x^4				1	99	2641	40444
x^5					1	219	10204
x^6						1	466
x^7							1

and cardinality n to the barycentric subdivision of the lattice of flats of $\mathsf{U}_{k,n}$. This can be combined with some of our formulas for $\underline{\mathrm{H}}_{\mathsf{M}}(x)$ and $\mathrm{H}_{\mathsf{M}}(x)$, which show that these two polynomials can be written as a sum of polynomials with positive coefficients indexed by chains of flats.

Acknowledgements

The first author wants to thank Botong Wang for sharing some useful comments and ideas that drew his attention to the Hilbert–Poincaré series of matroids. The authors thank the organizers of the conference "Geometry meets combinatorics in Bielefeld" for setting up the conference in which the authors met each other and where this collaboration started.

References

[1] K. Adiprasito, J. Huh, and E. Katz. "Hodge theory for combinatorial geometries". *Ann. of Math. (2)* **188**.2 (2018), pp. 381–452. DOI.

[2] F. Ardila and M. Sanchez. "Valuations and the Hopf Monoid of Generalized Permutahedra". *Int. Math. Res. Not. IMRN*, rnab355 (Jan. 2022). rnab355. DOI.

[3] C. A. Athanasiadis. "Gamma-positivity in combinatorics and geometry". *Sém. Lothar. Combin.* **77** ([2016–2018]), Art. B77i, 64.

[4] C. A. Athanasiadis. "Binomial Eulerian polynomials for colored permutations". *J. Combin. Theory Ser. A* **173** (2020), pp. 105214, 38. DOI.

[5] N. Bansal, R. A. Pendavingh, and J. G. van der Pol. "On the number of matroids". *Combinatorica* **35**.3 (2015), pp. 253–277. DOI.

[6] T. Braden, J. Huh, J. P. Matherne, N. Proudfoot, and B. Wang. "A semi-small decomposition of the Chow ring of a matroid". *Adv. Math.* **409** (2022), Paper No. 108646. DOI.

[7] T. Braden, J. Huh, J. P. Matherne, N. Proudfoot, and B. Wang. "Singular Hodge theory for combinatorial geometries". Oct. 2022. arXiv:2010.06088.

[8] T. Braden and A. Vysogorets. "Kazhdan-Lusztig polynomials of matroids under deletion". *Electron. J. Combin.* **27**.1 (2020), Paper No. 1.17, 17.

[9] P. Brändén. "Unimodality, log-concavity, real-rootedness and beyond". *Handbook of enumerative combinatorics*. Discrete Math. Appl. (Boca Raton). CRC Press, Boca Raton, FL, 2015, pp. 437–483.

[10] P. Brändén and K. Jochemko. "The Eulerian transformation". *Trans. Amer. Math. Soc.* **375**.3 (2022), pp. 1917–1931. DOI.

[11] B. Elias, N. Proudfoot, and M. Wakefield. "The Kazhdan-Lusztig polynomial of a matroid". *Adv. Math.* **299** (2016), pp. 36–70. DOI.

[12] E. M. Feichtner and S. Yuzvinsky. "Chow rings of toric varieties defined by atomic lattices". *Invent. Math.* **155**.3 (2004), pp. 515–536. DOI.

[13] L. Ferroni, G. D. Nasr, and L. Vecchi. "Stressed hyperplanes and Kazhdan-Lusztig gamma-positivity for matroids". *Int. Math. Res. Not. IMRN* (Oct. 2022). rnac270. DOI.

[14] L. Ferroni and B. Schröter. "Valuative invariants for large classes of matroids". Aug. 2022. arXiv:2208.04893.

[15] A. L. L. Gao, L. Lu, M. H. Y. Xie, A. L. B. Yang, and P. B. Zhang. "The Kazhdan-Lusztig polynomials of uniform matroids". *Adv. in Appl. Math.* **122** (2021), Paper No. 102117, 24. DOI.

[16] K. Gedeon, N. Proudfoot, and B. Young. "Kazhdan-Lusztig polynomials of matroids: a survey of results and conjectures". *Sém. Lothar. Combin.* **78B** (2017), Art. 80, 12.

[17] J. Haglund and P. B. Zhang. "Real-rootedness of variations of Eulerian polynomials". *Adv. in Appl. Math.* **109** (2019), pp. 38–54. DOI.

[18] T. Hameister, S. Rao, and C. Simpson. "Chow rings of vector space matroids". *J. Comb.* **12**.1 (2021), pp. 55–83. DOI.

[19] B. Han. "Gamma-positivity of derangement polynomials and binomial Eulerian polynomials for colored permutations". *J. Combin. Theory Ser. A* **182** (2021), Paper No. 105459, 22. DOI.

[20] K. Lee, G. D. Nasr, and J. Radcliffe. "A Combinatorial Formula for Kazhdan-Lusztig Polynomials of Sparse Paving Matroids". *Electron. J. Combin.* **28**.4 (2021), Paper No. 4.44–. DOI.

[21] D. Mayhew, M. Newman, D. Welsh, and G. Whittle. "On the asymptotic proportion of connected matroids". *European J. Combin.* **32**.6 (2011), pp. 882–890. DOI.

[22] A. Postnikov, V. Reiner, and L. Williams. "Faces of generalized permutohedra". *Doc. Math.* **13** (2008), pp. 207–273.

[23] N. Proudfoot, Y. Xu, and B. Young. "The Z-polynomial of a matroid". *Electron. J. Combin.* **25**.1 (2018), Paper No. 1.26, 21.

[24] J. Shareshian and M. L. Wachs. "Gamma-positivity of variations of Eulerian polynomials". *J. Comb.* **11**.1 (2020), pp. 1–33. DOI.

[25] M. Stevens. *Real-rootedness conjectures in matroid theory*. Thesis (Bachelor)–Stanford University. 2021, p. 27. Link.