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Counting linear extensions of posets with determinants of hook lengths

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Abstract. We introduce a class of posets, called mobile posets, which includes both ribbon posets and d-complete posets, whose number of linear extensions is given by a determinant of a matrix where entries are products of hook lengths. We also give q-analogs of this determinantal formula in terms of the inversion statistic.

Résumé. Nous introduisons une classe d'ensembles partiellement ordonnés, appelés ensembles mobiles partiellement ordonnés, qui comprend à la fois des ensembles partiellement ordonnés en ruban et des ensembles partiellement ordonnés complets d, dont le nombre d'extensions linéaires est donné par un déterminant d'une matrice où les entrées sont des produits de longueurs de crochet. Nous donnons également des q-analogues de cette formule déterminante en termes de statistique d'inversion.

Keywords: poset, linear extension, d-complete poset

1 Introduction

Linear extensions of posets are fundamental objects in combinatorics and computer science. The number of linear extensions of a poset $\mathcal{P}$, denoted by $e(\mathcal{P})$, is a measure of the complexity of the poset. However, computing $e(\mathcal{P})$ is a difficult problem—it is #P-complete [4], even for posets with restricted height or dimension [5]. Fortunately, for some posets that appear in algebraic and enumerative combinatorics, their number of

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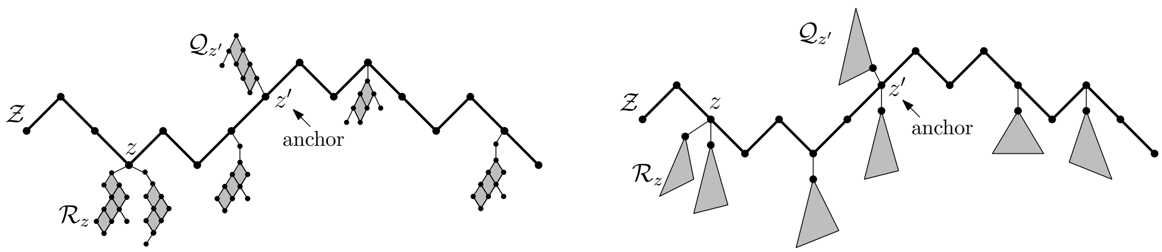

Figure 1: Left: schematic of a mobile poset. Right: schematic of a mobile tree poset. The shaded rhombi depict d-complete posets, and the shaded triangles depict rooted tree posets.

linear extensions can be efficiently computed through product formulas (posets arising from Young diagrams [6], rooted tree posets [8], d-complete posets [12]), determinants (posets arising from skew Young diagrams [1]), or recursive algorithms (tree posets [2]).

The main result of this paper is to give a determinantal formula for the number of linear extensions of *mobile posets*, a class of posets which includes both ribbons and d-complete posets. A mobile poset[1] $\mathcal{P}$ is a poset obtained from a ribbon poset $\mathcal{Z}$ by allowing every element z in $\mathcal{Z}$ to cover the maximal element of a nonnegative number of disjoint d-complete posets, and by letting at most one element z' of $\mathcal{Z}$ be covered by a certain element of a d-complete poset (see Figure 1: Left). If the d-complete posets in this description are restricted to rooted tree posets, then the posets in the resulting family are called *mobile tree posets* (see Figure 1: Right).

In Section 2, we review basic poset definitions including MacMahon's enumeration of the linear extensions of a ribbon poset. Section 3 presents the technique of folding posets which allows one to use inclusion-exclusion to enumerate linear extensions. Section 4 reviews the basics of d-complete posets and their hook length formulas. In Section 5, we introduce the component tree of a poset, which allows us to state our main theorem for mobile posets (Theorem 6.4) in Section 6. Lastly, we present a q-analog of our main theorem (Theorem 7.3) for the case of mobile tree posets, using inversions. In the full version of our paper [7], we present an additional q-analog, which uses major index.

2 Preliminaries

A *partially-ordered set* (poset) is a pair $(\mathcal{P}, \leq_{\mathcal{P}})$, with $\mathcal{P}$ a finite set and $\leq_{\mathcal{P}}$ a binary relation on $\mathcal{P}$ that is reflexive, antisymmetric, and transitive. We denote a poset by its underlying set when the order relation is clear from context. Throughout, we view $\leq_{\mathcal{P}}$ as

[1]The name "mobile" was chosen for the poset's resemblance to mobiles for babies and to the kinetic sculptures of Alexander Calder.

both a subset of $\mathcal{P}^2$ and as a way to compare two elements of $\mathcal{P}$, depending on context. (Thus, writing $(x,y) \in \leq_{\mathcal{P}}$ is equivalent to writing $x \leq_{\mathcal{P}} y$.) We denote the set of cover relations of $\mathcal{P}$ by $\lessdot_{\mathcal{P}}$. An *(induced) subposet* $\mathcal{Q}$ of $\mathcal{P}$ is a poset whose underlying set is a subset of the elements of $\mathcal{P}$, and whose relations are given by $s \leq_{\mathcal{Q}} t$ if and only if $s \leq_{\mathcal{P}} t$. Given two elements $x, y \in \mathcal{P}$, the *interval* $[x,y]$ is the subposet $\{z \in \mathcal{P} \mid x \leq_{\mathcal{P}} z \leq_{\mathcal{P}} y\}$.

If $\mathcal{P}$ and $\mathcal{Q}$ are two posets, we define their *disjoint sum* $\mathcal{P} + \mathcal{Q}$ as the poset with underlying set the disjoint union $\mathcal{P} \sqcup \mathcal{Q}$ and with relations the disjoint union $\leq_{\mathcal{P}} \sqcup \leq_{\mathcal{Q}}$. If $E \subset \mathcal{P}$, we denote by $\mathcal{P} \setminus E$ the poset with underlying set $\mathcal{P} \setminus E$ and with relations $\leq_{\mathcal{P}\setminus E} := \leq_{\mathcal{P}} \setminus \{(x,y) \in \leq_{\mathcal{P}} \mid x \in E \text{ or } y \in E\}$. Given a poset $\mathcal{P}$ with two incomparable elements x and y, let $\mathcal{P} \oplus \{(x,y)\}$ be the poset obtained by adding the cover relation (x,y) and taking the transitive closure.

Definition 2.1. *Let $\mathcal{P}, \mathcal{Q}_1, \ldots, \mathcal{Q}_m$ be disjoint posets, let p be an element in $\mathcal{P}$, and let q_i be an element in $\mathcal{Q}_i$ for $i = 1, \ldots, m$. The* slant sum *of $\mathcal{P}, \mathcal{Q}_1, \ldots, \mathcal{Q}_m$ at p and $q_1, \ldots, q_m$ is the poset*

$$\mathcal{P} \underset{i=1,\ldots,m}{{}^{p}\backslash_{q_i}} \mathcal{Q}_i := (\mathcal{P} + \mathcal{Q}_1 + \cdots + \mathcal{Q}_m) \oplus \{(q_1, p), \ldots, (q_m, p)\}.$$

The slant sum operation above is associative with fixed p, so this construction does not depend on the order in which we add the posets $\mathcal{Q}_i$ to $\mathcal{P}$.

We now introduce the main object of study in this paper: linear extensions of posets.

Definition 2.2. *A* linear extension *of an n-element poset $\mathcal{P}$ is a bijection $f : \mathcal{P} \to [n]$ that is order-preserving; that is, if $x \leq_{\mathcal{P}} y$, then $f(x) \leq f(y)$. We denote by $\mathscr{L}(\mathcal{P})$ the set of all linear extensions of $\mathcal{P}$ and by $e(\mathcal{P}) := \#\mathscr{L}(\mathcal{P})$ the number of linear extensions of $\mathcal{P}$.*

An important class of posets to which our theory applies is the class of ribbon posets. Let $S = \{s_1, \ldots, s_k\} \subset [n-1]$ with $s_1 < \cdots < s_k$. A *ribbon* poset $\mathcal{Z}$ with descent set S is the poset with underlying set $\{z_1, \ldots, z_n\}$ whose cover relations are $z_{i+1} \lessdot z_i$ if $i \in S$ and $z_i \lessdot z_{i+1}$ if $i \notin S$. The following classical theorem gives a determinant formula for the linear extensions of ribbon posets.

Theorem 2.3 (MacMahon [9, vol. I, p.190])**.** *The number of linear extensions of a ribbon poset $\mathcal{Z}$ with n elements and descent set $S \subset [n-1]$ is given by*

$$e(\mathcal{Z}) = n! \cdot \det \left(\frac{1}{(s_{j+1} - s_i)!} \right)_{0 \leq i,j \leq k}, \tag{2.1}$$

where $s_0 = 0$ and $s_{k+1} = n$.

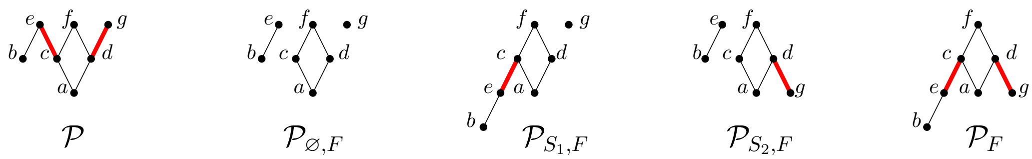

Figure 2: Examples of a fold of $\mathcal{P}$ and partial folds where $F = \{(c,e),(d,g)\}$, $S_1 = \{(c,e)\}$, and $S_2 = \{(d,g)\}$.

3 Folding and an alternating formula for linear extensions

We begin with a simple inclusion-exclusion formula for $e(\mathcal{P})$.

Definition 3.1. *Let $\mathcal{P}$ be a poset, $F \subset \lessdot_{\mathcal{P}}$, and $F^{\mathrm{op}} := \{(y,x) \in \mathcal{P}^2 \mid (x,y) \in F\}$. We write $\mathcal{P} \ominus F$ for the poset with the same underlying set as $\mathcal{P}$, but with cover relations $\lessdot_{\mathcal{P}\setminus F} := \lessdot_{\mathcal{P}} \setminus F$. We call a* fold *of $\mathcal{P}$ at F the poset*

$$\mathcal{P}_F := (\mathcal{P} \ominus F) \oplus F^{\mathrm{op}}$$

obtained by deleting the cover relations in F, adding the opposite cover relations, and taking the transitive closure. If $S \subset F$, then we call a partial fold *of $\mathcal{P}$ at S the poset*

$$\mathcal{P}_{S,F} := (\mathcal{P} \ominus F) \oplus S^{\mathrm{op}}.$$

Example 3.2. *Consider the seven element poset $\mathcal{P}$ in the left of Figure 2. Let $F = \{(c,e),(d,g)\}$, $S_1 = \{(c,e)\}$, and $S_2 = \{(d,g)\}$. The posets $\mathcal{P}_{\varnothing,F}$, $\mathcal{P}_{S_1,F}$, $\mathcal{P}_{S_2,F}$, and $\mathcal{P}_F$ are also depicted in Figure 2.*

The next lemma describes how the number of linear extensions of a poset changes when folding at a single cover relation.

Lemma 3.3. *Let $\mathcal{P}$ be a poset and (x,y) be in $\lessdot_{\mathcal{P}}$. Then*

$$\mathscr{L}(\mathcal{P}) = \mathscr{L}(\mathcal{P} \ominus \{(x,y)\}) \setminus \mathscr{L}(\mathcal{P}_{\{(x,y)\}}). \tag{3.1}$$

In particular, we have that

$$e(\mathcal{P}) = e(\mathcal{P} \ominus \{(x,y)\}) - e(\mathcal{P}_{\{(x,y)\}}).$$

Example 3.4. *Consider the seven element poset $\mathcal{P}$ in Figure 3: Left. Choosing either the cover relation (c,e) or (a,c), we obtain*

$$\begin{aligned} 77 = e(\mathcal{P}) &= e(\mathcal{P} \ominus \{(c,e)\}) - e(\mathcal{P}_{\{(c,e)\}}) = 105 - 28, && (3.2a)\\ &= e(\mathcal{P} \ominus \{(a,c)\}) - e(\mathcal{P}_{\{(a,c)\}}) = 117 - 40. && (3.2b)\end{aligned}$$

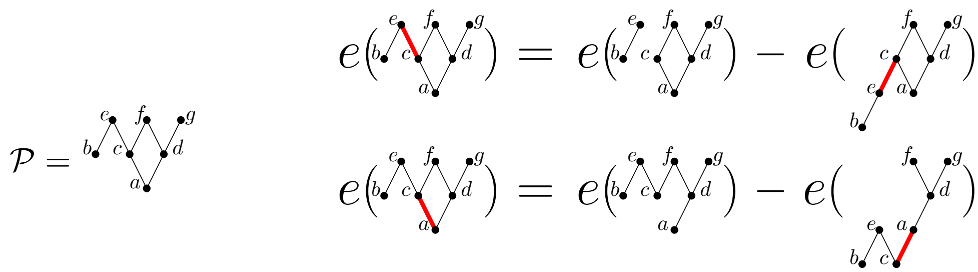

Figure 3: Example of using the inclusion-exclusion formula from Lemma 3.3 to calculate $e(\mathcal{P})$.

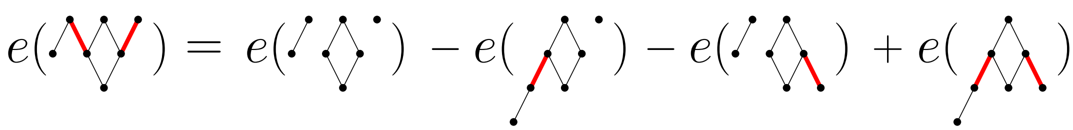

Figure 4: An example of the linear extension formula from (3.3).

See the right image in Figure 3 for an illustration of these inclusion-exclusion formulas. As a further illustration of (3.2a)*, consider the following linear extension $\eta \in \mathscr{L}(\mathcal{P})$:*

$$a \mapsto 1, \quad b \mapsto 2, \quad c \mapsto 3, \quad d \mapsto 4, \quad e \mapsto 5, \quad f \mapsto 6, \quad g \mapsto 7.$$

Notice that $\eta \in \mathscr{L}(\mathcal{P} \ominus \{(c,e)\})$ and $\eta \notin \mathscr{L}(\mathcal{P}_{\{(c,e)\}})$, since $\eta(c) < \eta(e)$.

The next result follows from repeatedly applying Lemma 3.3.

Corollary 3.5. *Let $\mathcal{P}$ be a poset, and let $F \subset \lessdot_{\mathcal{P}}$. Then*

$$e(\mathcal{P}) = \sum_{S \subset F} (-1)^{\#S} e(\mathcal{P}_{S,F}). \tag{3.3}$$

Example 3.6. *For the poset $\mathcal{P}$ from Example 3.4, Corollary 3.5 yields the formula shown in Figure 4 when $F = \{(c,e),(d,g)\}$ is the set of cover relations depicted in red on the left-hand side of the above equation.*

4 d-complete posets

The class of d-complete posets is an important family of posets whose linear extensions we will enumerate. Defined by Proctor in [11], d-complete posets form a large class of posets containing rooted tree posets and posets arising from Young diagrams,[2] while still retaining a hook-length formula for their number of linear extensions. We recall their definition below (see Definition 4.1).

A poset $\mathcal{P}$ has a *diamond* if there are four elements w, x, y, z in $\mathcal{P}$ such that z covers x and y, while x and y cover w. For $k \geq 3$, a *double-tailed diamond poset* d_k is a poset obtained by adding a $k-3$ chain to the top and bottom of a diamond (w, x, y, z). The *neck* elements are the $k-2$ elements above the two incomparable elements x and y. A d_k*-interval* is an interval $[u, v]$ which is isomorphic to d_k.

A subset S of $\mathcal{P}$ is *convex* if for any $x, y \in S$ and any $z \in \mathcal{P}$ satisfying $x \leq z \leq y$, one has that $z \in S$. For $k \geq 3$, a d_k^-*-convex set* is a convex set of $\mathcal{P}$ that is isomorphic to a d_k-interval with the maximal element removed. Note that for $k \geq 4$, a d_k^--convex set is an interval.

Definition 4.1 ([11])**.** *A poset $\mathcal{P}$ is d-*complete *if, for any $k \geq 3$, the following properties are satisfied:*

1. *If I is a d_k^--convex set, then there exists an element p in $\mathcal{P}$ that covers the maximal elements of I.*
2. *If $[w, z]$ is a d_k-interval, then z does not cover any elements of $\mathcal{P}$ outside $[w, z]$.*
3. *There are no d_k^--convex sets which differ only in their minimal elements.*

A connected d-complete poset has a unique maximal element [10, Section 14]. Given a connected d-complete poset $\mathcal{P}$, its *top tree* Γ is the (induced) subgraph of the Hasse diagram of $\mathcal{P}$ consisting of vertices x in $\mathcal{P}$ such that $y \geq_{\mathcal{P}} x$ is covered by at most one other element. (This subgraph is indeed a tree.) An element y of $\mathcal{P}$ is *acyclic* if $y \in \Gamma$ and is not part of the neck of any d_k-interval of $\mathcal{P}$. Note that if $\mathcal{P}$ is a rooted tree, then $\Gamma = \mathcal{P}$ and all its elements are acyclic.

Slant sums (see Definition 2.1) can be used to combine two d-complete posets to obtain a larger d-complete poset.

Proposition 4.2 (Proctor [10, Proposition B])**.** *Let $\mathcal{P}_1$ be a connected d-complete poset with an acyclic element y, and let $\mathcal{P}_2$ be a connected d-complete poset with maximal element x. Then the slant sum $\mathcal{P} := \mathcal{P}_1{}^y\backslash_x \mathcal{P}_2$ is a d-complete poset, and the acyclic elements of $\mathcal{P}_1$ and $\mathcal{P}_2$ are acyclic elements of $\mathcal{P}$.*

Next, we recall the hook-length formula for the number of linear extensions of a d-complete poset.

Definition 4.3 ([11])**.** *The* hook length $h_{\mathcal{P}}(z)$ *of an element z in a d-complete poset $\mathcal{P}$ is defined as follows:*

[2] See [10, Table 1] for a complete classification of d-complete posets.

1. *If z is not in the neck of any d_k-interval, then $h_{\mathcal{P}}(z) = \#\{y \mid y \leq_{\mathcal{P}} z\}$.*
2. *If z is in the neck of a d_k-interval, then we can find some element w such that $[w, z]$ is a d_ℓ-interval for some ℓ. If x and y are the two incomparable elements in the d_ℓ-interval, then $h_{\mathcal{P}}(z) = h_{\mathcal{P}}(x) + h_{\mathcal{P}}(y) - h_{\mathcal{P}}(w)$.*

Theorem 4.4 (Peterson–Proctor [12]). *The number of linear extensions of a d-complete poset $\mathcal{P}$ with n elements is*

$$e(\mathcal{P}) = \frac{n!}{\prod_{x \in \mathcal{P}} h_{\mathcal{P}}(x)},$$

where $h_{\mathcal{P}}(x)$ is the hook length of x in $\mathcal{P}$ from Definition 4.3.

5 Component trees and component arrays

For the rest of the paper, we assume that $\mathcal{P}$ is connected. However, our results can easily be adapted to the case where $\mathcal{P}$ is a disconnected poset.

Definition 5.1. *We define the* component tree *of $\mathcal{P}_F$, where $F \subset \lessdot_{\mathcal{P}}$, to be the tree $C(\mathcal{P}_F)$ with vertices $\{\sigma_0, \ldots, \sigma_k\}$ the connected components of the poset $\mathcal{P} \ominus F$ and edges $\{\sigma_x, \sigma_y\}$ for all $(x, y) \in F$, where $x \in \sigma_x$ and $y \in \sigma_y$. That $C(\mathcal{P}_F)$ is a tree follows from the fact that none of the cover relations in F lie in an undirected cycle in the Hasse diagram of $\mathcal{P}$.*

Definition 5.2. *Suppose $\#F = k$ and $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_k)$ is a total order on the vertices of $C(\mathcal{P}_F)$. The* component array *$M_\sigma(\mathcal{P}_F)$ is the triangular array of posets*

$$(M_\sigma(\mathcal{P}_F))_{i,j} := C(\mathcal{P}_F)[i, j],$$

where $0 \leq i \leq j \leq k$ and $C(\mathcal{P}_F)[i, j]$ is the subposet of $\mathcal{P}_F$ on the elements in the connected components $\sigma_i, \sigma_{i+1}, \ldots, \sigma_j$ of $\mathcal{P} \ominus F$. We say σ is a path order *if each entry of the component array $M_\sigma(\mathcal{P}_F)$ is a connected poset.*

Example 5.3. *For the poset $\mathcal{P}$ and folds F from Example 3.6, Figure 5 depicts the component tree and component arrays for a path order $\sigma = (\sigma_0, \sigma_1, \sigma_2)$ and an order $\tau = (\sigma_1, \sigma_0, \sigma_2)$ that is not a path order.*

Proposition 5.4. *There is a path order σ on the vertices of $C(\mathcal{P}_F)$ if and only if $C(\mathcal{P}_F)$ is a path.*

6 Determinant formulas for linear extensions

In this section, we present our main theorem. This theorem gives a determinantal formula for the linear extensions of mobile posets. The definition of the latter appears below.

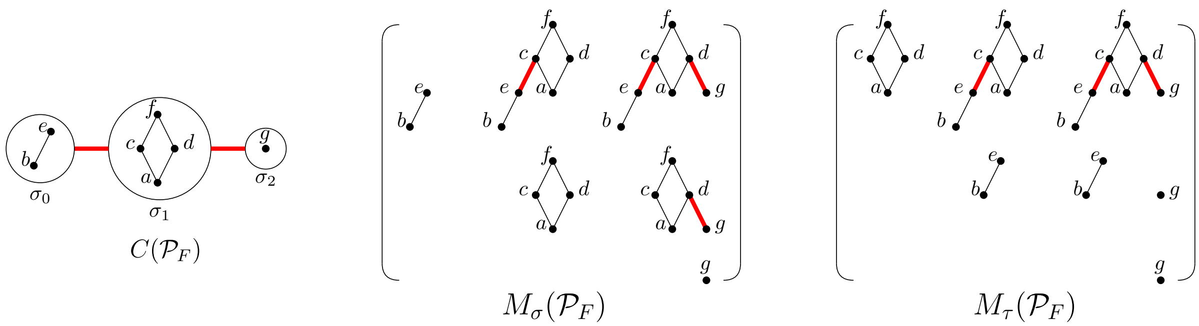

Figure 5: Left: example of component tree $C(\mathcal{P}_F)$ with a chosen total order $\sigma = (\sigma_0, \sigma_1, \sigma_2)$ on the vertices. Center: example of component array $M_\sigma(\mathcal{P}_F)$. Right: another example of a component array $M_\tau(\mathcal{P}_F)$ with a different order $\tau = (\sigma_1, \sigma_0, \sigma_2)$.

Definition 6.1. *A* (connected) mobile poset $\mathcal{P}$ *is a poset obtained from a ribbon* $\mathcal{Z}$ *by the following two operations:*

(i) For every element $z \in \mathcal{Z}$*, perform a slant sum* $\mathcal{Z} \; {}^{z}\!\!\setminus_{r_i}{}_{i=1,\ldots,m_z} \; \mathcal{R}_z^{(i)}$ *with* $m_z \geq 0$ *connected d-complete posets* $\mathcal{R}_z^{(i)}$ *with respective maximal elements* r_i*. Denote the resulting poset by* $\mathcal{P}'$*.*

(ii) For at most one element $z' \in \mathcal{Z}$*, perform a slant sum* $\mathcal{Q}_{z'} \; {}^{q}\!\!\setminus_{z'} \mathcal{P}'$ *where* $\mathcal{Q}_{z'}$ *is a connected d-complete poset and* q *is one of its acyclic elements. Such an element* z' *is called an* anchor.

If no such element $z' \in \mathcal{Z}$ *is used in Operation (ii), we say that the mobile is* free-standing with respect to the ribbon $\mathcal{Z}$*. If each poset attached to* $\mathcal{Z}$ *as above is a rooted tree, we say that* $\mathcal{P}$ *is a* mobile tree poset*. Additionally, we say that a mobile poset* $\mathcal{P}$ *is* free-standing *if there exists a ribbon* $\mathcal{Z}$ *with respect to which* $\mathcal{P}$ *is free-standing.*

See the left image in Figure 1 for a schematic of a mobile poset.

Example 6.2. *Figure 6 shows four examples: a free-standing mobile, a mobile, and two posets that cannot be expressed as mobiles.*

To state Theorem 6.4, we identify special types of folds to apply to a mobile poset $\mathcal{P}$ called path folds.

Definition 6.3. *Let* $\mathcal{P}$ *be a mobile poset with a ribbon* $\mathcal{Z}$ *with descents* S*. The set of* path folds *for* $\mathcal{P}$ *(with respect to* $\mathcal{Z}$*) is defined as*

$$F = \begin{cases} \{(z_{i+1}, z_i) \mid i \in S\} & \text{if } \mathcal{P} \text{ is free-standing,} \\ \{(z_{i+1}, z_i) \mid i \in S, i < j\} \cup \{(z_i, z_{i+1}) \mid i \notin S, i \geq j\} & \text{otherwise,} \end{cases} \tag{6.1}$$

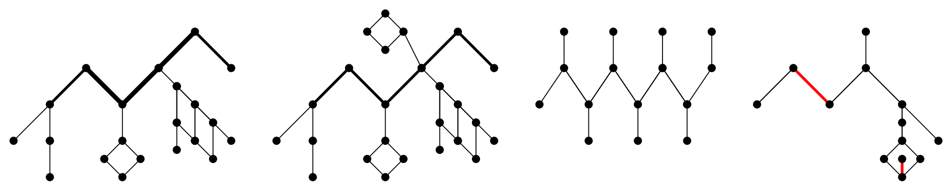

Figure 6: Examples of (a) a free-standing mobile poset, (b) a mobile poset, and (c) and (d) posets that are not mobile posets.

where j is the index of the anchor $z' = z_j$ covered by an acyclic element of a connected d-complete poset $\mathcal{Q}_{z'}$ (see Definition 6.1 (ii)).

Theorem 6.4. *Let $\mathcal{P}$ be a mobile poset with n elements, F the set of path folds for $\mathcal{P}$, and σ a path order that is compatible with F. Then*

$$e(\mathcal{P}) \;=\; n! \cdot \det(M_{i,j})_{0\le i,j\le k}, \quad \textit{for} \quad M_{i,j} := \begin{cases} 0 & \textit{if } j < i-1, \\ 1 & \textit{if } j = i-1, \\ 1/\prod_{x\in\mathcal{P}_{i,j}} h_{\mathcal{P}_{i,j}}(x) & \textit{otherwise,} \end{cases} \tag{6.2}$$

where k is the size of F and $\mathcal{P}_{i,j}$ is the connected d-complete poset $(M_\sigma(\mathcal{P}_F))_{i,j}$.

Example 6.5. *Consider the mobile poset $\mathcal{P}$ and set $F = \{(e,b),(f,d)\}$ of path folds pictured in Figure 7: Left. The component tree $C(\mathcal{P}_F)$ and the component array $M_\sigma(\mathcal{P}_F)$ are pictured in Figure 7: Center, Right. Applying Theorem 6.4 to $\mathcal{P}$ gives the determinantal formula*

$$e(\mathcal{P}) = 10! \cdot \det \begin{pmatrix} \frac{1}{1} & \frac{1}{9\cdot8\cdot5\cdot3\cdot2\cdot2\cdot2} & \frac{1}{10\cdot9\cdot6\cdot3\cdot2\cdot2\cdot2} \\ 1 & \frac{1}{8\cdot7\cdot5\cdot3\cdot2\cdot2} & \frac{1}{9\cdot8\cdot6\cdot3\cdot2\cdot2} \\ 0 & 1 & \frac{1}{1} \end{pmatrix} = 240.$$

7 Determinant formulas for q-analogs of linear extensions

A *labeled poset* $(\mathcal{P},\omega)$ is a poset $\mathcal{P}$ with n elements, together with a bijection $\omega\colon \mathcal{P} \to [n]$. A labeling ω is *regular* if we have the following: for all $x <_{\mathcal{P}} z$ and $y \in \mathcal{P}$, if $\omega(x) < \omega(y) < \omega(z)$ or $\omega(x) > \omega(y) > \omega(z)$ then $x <_{\mathcal{P}} y$ or $y <_{\mathcal{P}} z$. For more on regular labelings, we refer the reader to [3]. Additionally, we define $\operatorname{inv}(\mathcal{P},\omega)$ to be the number of *inversions* of $(\mathcal{P},\omega)$: pairs (x,y) with $\omega(x) > \omega(y)$ and $x <_{\mathcal{P}} y$.

Definition 7.1. *Let $(\mathcal{P},\omega)$ be a labeled poset. If $f\colon \mathcal{P} \to [n]$ is a linear extension of $\mathcal{P}$, then the permutation $\omega \circ f^{-1} \in \mathfrak{S}_n$ is called a* linear extension of the labeled poset $(\mathcal{P},\omega)$. *We write $\mathscr{L}(\mathcal{P},\omega)$ for the set of all linear extensions of $(\mathcal{P},\omega)$.*

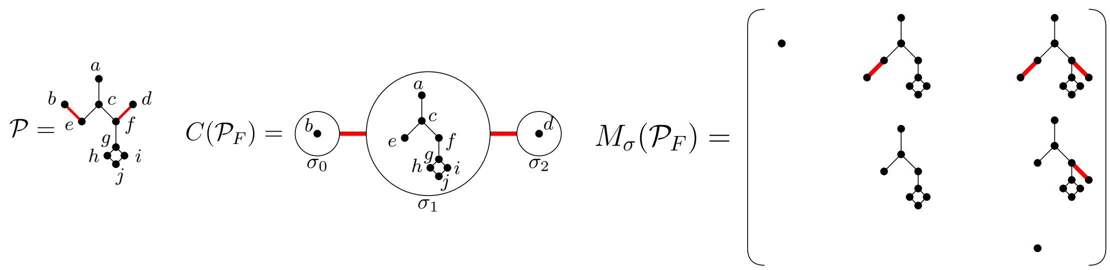

Figure 7: Left: a mobile poset $\mathcal{P}$ with folds highlighted. Center: its component tree $C(\mathcal{P}_F)$, with path order $\sigma = (\sigma_0, \sigma_1, \sigma_2)$. Right: its component array $M_\sigma(\mathcal{P}_F)$.

By restricting to the class of mobile tree posets $\mathcal{P}$, we identify a distinguished labeling of $\mathcal{P}$ defined below. With this labeling, we state our final result.

Definition 7.2. *Let $\mathcal{P}$ be a mobile tree poset, F the set of path folds for $\mathcal{P}$ and σ a path order compatible with F. Then σ gives an order $\mathcal{P}_{\sigma_0}, \mathcal{P}_{\sigma_1}, \dots, \mathcal{P}_{\sigma_k}$ on the connected components of the poset $\mathcal{P} \ominus F$. A labeling ω on $\mathcal{P}$ is called a σ-*partitioned labeling *if whenever $\sigma_i < \sigma_j$, we have*

$$\omega(x) < \omega(y) \qquad \textit{for every} \qquad x \in \mathcal{P}_{\sigma_i},\ y \in \mathcal{P}_{\sigma_j}.$$

*Moreover, ω is called a σ-*partitioned regular labeling *if it is a σ-partitioned labeling such that the restriction of ω to each connected component $\mathcal{P}_{\sigma_i}$ of $\mathcal{P} \ominus F$ is a regular labeling of that component.*

Theorem 7.3. *Let $(\mathcal{P}, \omega)$ be a labeled mobile tree poset with n elements, F the set of path folds for $\mathcal{P}$, σ a path order compatible with F, and ω a σ-partitioned regular labeling of $\mathcal{P}$. Then*

$$e_q^{inv}(\mathcal{P}, \omega) \;=\; [n]_q! \cdot \det(M_{i,j})_{0 \le i,j \le k}, \quad \textit{for} \quad M_{i,j} := \begin{cases} 0 & \textit{if } j < i-1, \\ 1 & \textit{if } j = i-1, \\ \dfrac{q^{inv(\mathcal{P}_{i,j}, \omega_{i,j})}}{\prod_{x \in \mathcal{P}_{i,j}} [h_{\mathcal{P}_{i,j}}(x)]_q} & \textit{otherwise}, \end{cases} \tag{7.1}$$

where k is the size of F and $(\mathcal{P}_{i,j}, \omega_{i,j})$ is the labeled rooted tree poset $(M_\sigma(\mathcal{P}_F, \omega))_{i,j}$.

Example 7.4. *Let $\mathcal{P}$ be the mobile tree poset in Figure 8 with σ-partitioned regular labeling ω given by $a \mapsto 1, b \mapsto 3, c \mapsto 6, d \mapsto 4, e \mapsto 2$, and $f \mapsto 5$. Applying Theorem 7.3 to $(\mathcal{P}, \omega)$ yields*

$$e_q^{inv}(\mathcal{P}, \omega) = [6]_q! \cdot \det \begin{pmatrix} \frac{1}{[1]_q} & \frac{q^3}{[5]_q[4]_q} & \frac{q^5}{[6]_q[5]_q} \\ 1 & \frac{q^3}{[4]_q[3]_q} & \frac{q^5}{[5]_q[4]_q} \\ 0 & 1 & \frac{1}{[1]_q} \end{pmatrix} = q^{10} + 3q^9 + 4q^8 + 3q^7 + q^6.$$

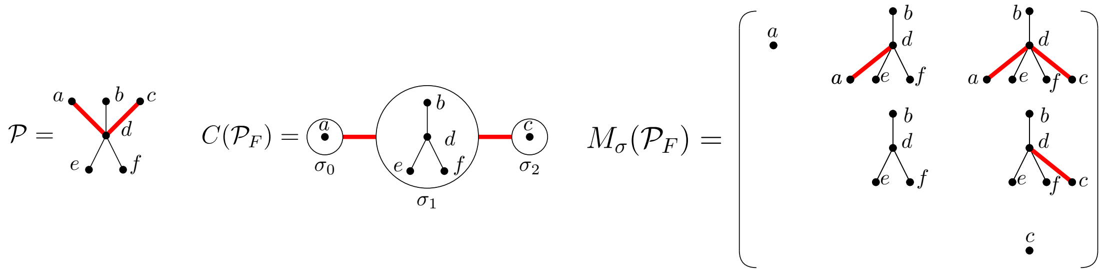

Figure 8: Left: a mobile tree poset $\mathcal{P}$ with folds highlighted. Center: its component tree $C(\mathcal{P}_F)$, with path order $\sigma = (\sigma_0, \sigma_1, \sigma_2)$. Right: its component array $M_\sigma(\mathcal{P}_F)$.

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