A semi-small decomposition of the Chow ring of a matroid

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$\boxed{\text{I.}}$ Setting

M matroid $\xrightarrow{\text{[Feichtner–Yuzvinsky '04]}}$ $\underline{CH}(M)$ – Chow ring of M

*[Adiprasito–Huh–Katz '18] write $A(M)$

– If M comes from a hyp. arr A in V (M realizable), then $\underline{CH}(M) \cong H^\cdot(X_V)$

↑ sm. proj var.

$\underline{\text{Thm}}$ [AHK '18]

~~$H^\cdot(X_V)$~~ $\underline{CH}(M)$ satisfies

- Poincaré duality (PD)
- Hard Lefschetz thm (HL)
- Hodge-Riemann rel'ns (HR)

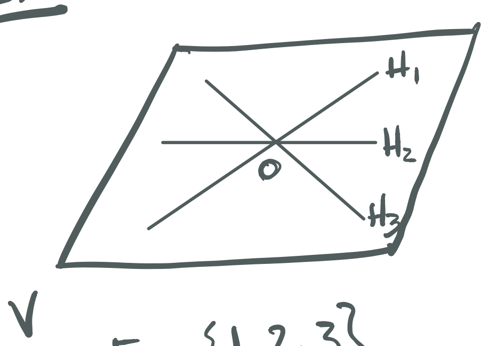
Key ingredient in pf of various log-concavity conj's for M.

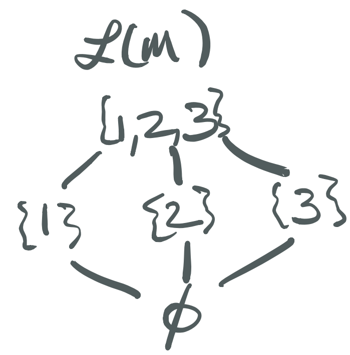
$\boxed{\text{II.}}$ Chow rings of matroids & the semi-small decomp.

M matroid

- $\mathcal{L}(M)$ lattice of flats

$\underline{\text{Ex}}$:





$E = \{1, 2, 3\}$

$$\underline{CH}(M) := \frac{\mathbb{Q}[x_F \mid F \in \mathcal{L}(M) \setminus \{E, \emptyset\}]}{\begin{pmatrix} x_{F_1} x_{F_2} & \text{if } F_1, F_2 \text{ incomp} \\ \sum_{i \in F} x_F - \sum_{j \in F} x_F & \forall \text{ distinct } i, j \in E \end{pmatrix}}$$

- If $rk(M) = d$, then $\underline{CH}^k(M) = 0$ if $k > d-1$.

- degree map

$$\underline{\deg}_M : \underline{CH}^{d-1}(M) \xrightarrow{\sim} \mathbb{Q}$$
$$x_{F_1} x_{F_2} \cdots x_{F_{d-1}} \longmapsto 1$$

$F_1 < F_2 < \cdots < F_{d-1}$
complete flag

Poincaré pairing: $\forall k \in \mathbb{Z}_{\geq 0}$,

$$\langle -, - \rangle_M : \underline{CH}^k(M) \times \underline{CH}^{d-k-1}(M) \longrightarrow \mathbb{Q}$$
$$(\eta_1, \eta_2) \longmapsto \underline{\deg}_M(\eta_1 \eta_2).$$

<u>Thm</u> [AHK '18]

Let $\underline{\ell} \in \underline{CH}^1(M)$ be an ample class. Then $\underline{CH}(M)$ satisfies

- (PD): $\forall k < \frac{d}{2}$, $\langle -, - \rangle_M$ is non-degenerate.
- (HL): $\forall k < \frac{d}{2}$,

$$\underline{CH}^k(M) \longrightarrow \underline{CH}^{d-k-1}(M)$$
$$\eta \longmapsto \underline{\ell}^{d-2k-1} \eta$$

is an isom.

- (HR): $\forall k < \frac{d}{2}$,

$$\underline{CH}^k(M) \times \underline{CH}^k(M) \longrightarrow \mathbb{Q}$$
$$(\eta_1, \eta_2) \longmapsto (-1)^k \underline{\deg}_M(\underline{\ell}^{d-2k-1} \eta_1 \eta_2)$$

is pos. def. on $\ker(\underline{\ell}^{d-2k})$.

Main Thm [Braden–Huh–M.–Proudfoot–Wang '20]

Let $i \in E$ **not** be a coloop. Then,

$$\underline{CH}(M) = \underline{\Theta}_i(\underline{CH}(M\setminus i)) \oplus \bigoplus_{F \in \underline{S}_i} x_{F\cup i}\, \underline{\Theta}_i(\underline{CH}(M\setminus i)).$$

? ?

- indecomposable gdd $\underline{CH}(M\setminus i)$-mods
- pairwise $\perp$ wrt $\langle -,- \rangle_M$

gdd alg. hom.

$$\underline{\Theta}_i : \underline{CH}(M\setminus i) \longrightarrow \underline{CH}(M)$$

$$x_F \longmapsto x_F + x_{F\cup i}$$

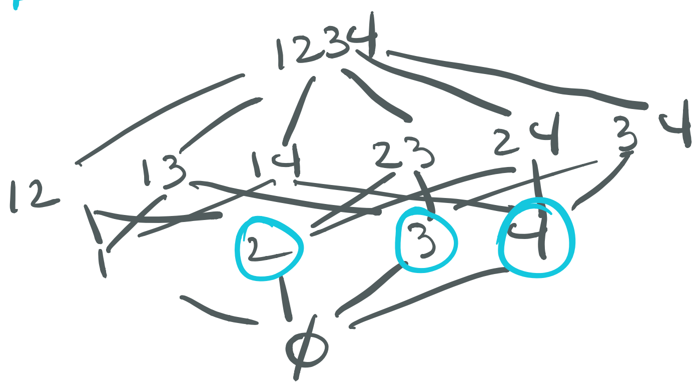
0 if label is not a flat in $\mathcal{L}(M)$.

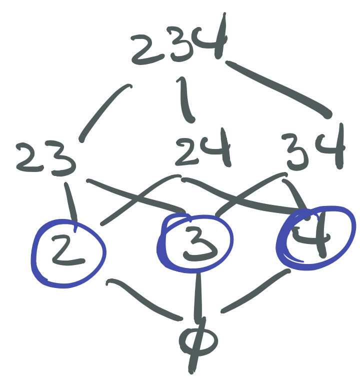
$$\underline{S}_i = \left\{ F \;\middle|\; \begin{array}{l} \cdot\ \emptyset \neq F \subsetneq E\setminus i \\ \cdot\ F \in \mathcal{L}(M) \\ \cdot\ F\cup i \in \mathcal{L}(M) \end{array} \right\}$$

both terms $\neq 0$.

Ex: $M = U_{3,4}$ (4 hyperplanes in $\mathbb{C}^3$ w/ int. $\{0\}$) $M\setminus 1 = B_3$

$E\setminus 1 = \{2,3,4\}$





$$x_2 \longmapsto x_2 + x_{12}$$

$$x_{23} \longmapsto x_{23} + \cancel{x_{123}} \to 0$$

Main Thm [Braden-Huh-M.-Proudfoot-Wang '20]
Let $i \in E$ not be a coloop. Then,

$$\underline{CH}(M) = \underline{\Theta}_i(\underline{CH}(M\backslash i)) \oplus \bigoplus_{F \in \underline{S}_i} x_{F\cup i}\,\underline{\Theta}_i(\underline{CH}(M\backslash i)).$$

Prep. Work:

- Understand the summands:

$(*)$ $x_{F\cup i}\,\underline{\Theta}_i(\underline{CH}(M\backslash i)) = \underline{\psi}^{F\cup i}(\underline{CH}(M_{F\cup i}) \otimes \underline{\Theta}_i^{F\cup i}\,\underline{CH}(M^F))$

$\underline{\psi}^{F\cup i} : \underline{CH}(M_{F\cup i}) \otimes \underline{CH}(M^{F\cup i}) \longrightarrow \underline{CH}(M)$

Gysin pushforward

$\underline{\Theta}_i^{F\cup i} : \underline{CH}(M^F) \longrightarrow \underline{CH}(M^{F\cup i})$

- Compatibility of maps w/ $\underline{\deg}_M$:

Similar cptibility here.

$(**)$ $\underline{\deg}_{M\backslash i} = \underline{\deg}_M \circ \underline{\Theta}_i$

If $\underline{CH}(M_F)$ & $\underline{CH}(M^F)$ have (PD), then $\underline{\psi}^F$ inj.

If $\underline{CH}(M\backslash i)$ has (PD), then $\underline{\Theta}_i$ inj.

- (PD) & semi-small decomp. (Pf. Sketch):
  - Ind. on $|E|$.
  - Show summands in RHS $\perp$ wrt $\langle -,- \rangle_M$
  - $(**)$ $\Longrightarrow$ $\langle -,- \rangle_M \big|_{\underline{\Theta}_i(\underline{CH}(M\backslash i))}$ is nondeg.
  - Similar for other summands
  - $\Rightarrow$ RHS is a $\oplus$ w/ nondeg. $\langle -,- \rangle_M \big|_{\text{RHS}}$.
  - Check RHS $= \underline{CH}(M)$ (deg. by deg).

- (HL) & (HR) are similar

$\boxed{\text{IV}}$ Why?

- Simpler pf of [AHK'18]

  They prove (PD), (HL), (HR) for many intermediate objs corresp. to blowups & must keep track of Kähler package at each step.

- Our main reason:

  Key ingredient in proof of:

  Thm [Braden-Huh-M.-Proudfoot-Wang]

  - Top-Heavy conj.
  - Non-negativity of coeffs of KL polys

  $\Big]$ $\forall$ matroids