COMBINATORICS OF EXCEPTIONAL SEQUENCES IN TYPE A

ALEXANDER GARVER, KIYOSHI IGUSA, JACOB P. MATHERNE, AND JONAH OSTROFF

Abstract. Exceptional sequences are certain ordered sequences of quiver representations. We introduce a class of objects called strand diagrams and use this model to classify exceptional sequences of representations of a quiver whose underlying graph is a type $\mathbb{A}_n$ Dynkin diagram. We also use variations of this model to classify $\mathbf{c}$-matrices of such quivers, to interpret exceptional sequences as linear extensions of posets, and to give a simple bijection between exceptional sequences and certain chains in the lattice of noncrossing partitions. This work extends a classification of exceptional sequences for the linearly-ordered quiver obtained in [GM15] by the first and third authors.

Contents

1. Introduction

Exceptional sequences are certain sequences of quiver representations with strong homological properties. They were introduced in [GR87] to study exceptional vector bundles on $\mathbb{P}^2$, and more recently, Crawley-Boevey showed that the braid group acts transitively on the set of complete exceptional sequences (exceptional sequences of maximal length) [CB93]. This result was generalized to hereditary Artin algebras by Ringel [Rin94]. Since that time, Meltzer has also studied exceptional sequences for weighted projective lines [Mel04], and Araya for Cohen-Macaulay modules over one dimensional graded Gorenstein rings with a simple singularity [Ara99]. Exceptional sequences have been shown to be related to many other areas of mathematics since their invention:

- chains in the lattice of noncrossing partitions [Bes03, HK13, IT09],
- $\mathbf{c}$-matrices and cluster algebras [ST13],
- factorizations of Coxeter elements [IS10], and
- t-structures and derived categories [Bez03, BK89, Rud90].

Despite their ubiquity, very little work has been done to concretely describe exceptional sequences, even for path algebras of Dynkin quivers [Ara13, GM15]. In this paper, we give a concrete description of exceptional sequences for type $\mathbb{A}_n$ quivers of any orientation. This work extends a classification of exceptional sequences for the linearly-ordered quiver obtained in [GM15] by the first and third authors.

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Exceptional sequences are composed of indecomposable representations which have a particularly nice description. For a quiver Q of type $\mathbb{A}_n$, the indecomposable representations are completely determined by their dimension vectors, which are of the form $(0,...,0,1,...,1,0,...,0)$. Let us denote such a representation by $X^{\epsilon}_{i,j}$, where ϵ is a vector that keeps track of the orientation of the quiver, and $i+1$ and j are the positions where the string of 1's begins and ends, respectively.

This simple description allows us to view exceptional sequences as combinatorial objects. Define a map Φ_ϵ which associates to each indecomposable representation $X^{\epsilon}_{i,j}$ a strand $\Phi_\epsilon(X^{\epsilon}_{i,j})$ on a collection of $n+1$ dots.

$X^{\epsilon}_{0,1}$ $\qquad \Bbbk \xleftarrow[-]{0} 0$

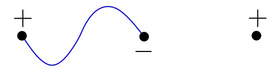

FIGURE 1. An example of the indecomposable representation $X^{\epsilon}_{0,1}$ on a type $\mathbb{A}_2$ quiver and the corresponding strand $\Phi_\epsilon(X^{\epsilon}_{0,1})$

As exceptional sequences are collections of representations, the map Φ_ϵ allows one to regard them as collections of strands. The following lemma is the foundation for all of our results in this paper (it characterizes the homological data encoded by a pair of strands and thus by a pair of representations). Since exceptional sequences are sequences of representations, each pair of which satisfy certain homological properties, Lemma 3.5 allows us to completely classify exceptional sequences using strand diagrams.

Lemma 3.5. Let Q_ϵ be given. Fix two distinct indecomposable representations $U, V \in \text{ind}(\text{rep}_{\Bbbk}(Q_\epsilon))$.

a) The strands $\Phi_\epsilon(U)$ and $\Phi_\epsilon(V)$ intersect nontrivially if and only if neither (U,V) nor (V,U) are exceptional pairs.

b) The strand $\Phi_\epsilon(U)$ is clockwise from $\Phi_\epsilon(V)$ if and only if (U,V) is an exceptional pair and (V,U) is not an exceptional pair.

c) The strands $\Phi_\epsilon(U)$ and $\Phi_\epsilon(V)$ do not intersect at any of their endpoints and they do not intersect nontrivially if and only if (U,V) and (V,U) are both exceptional pairs.

The paper is organized in the following way. In Section 2, we give the preliminaries on quivers and their representations which are needed for the rest of the paper.

In Section 3.1, we decorate our strand diagrams with strand-labelings and oriented edges so that they can keep track of both the ordering of the representations in a complete exceptional sequence as well as the signs of the rows in the **c**-matrix it came from. While unlabeled diagrams classify complete exceptional collections (Theorem 3.6), we show that the new decorated diagrams classify more complicated objects called exceptional sequences (Theorem 3.9). Although Lemma 3.5 is the main tool that allows us to obtain these results, we delay its proof to Section 3.2.

The work of Speyer and Thomas (see [ST13]) allows complete exceptional sequences to be obtained from **c**-matrices. In [ONA$^+$13], the number of complete exceptional sequences in type $\mathbb{A}_n$ is given, and there are more of these than there are **c**-matrices. Thus, it is natural to ask exactly which **c**-matrices appear as strand diagrams. By establishing a bijection between the mixed cobinary trees of Igusa and Ostroff [IO13] and a certain subcollection of strand diagrams, we give an answer to this question in Section 4.

In Section 5, we ask how many complete exceptional sequences can be formed using the representations in a complete exceptional collection. It turns out that two complete exceptional sequences can be formed in this way if they have the same underlying chord diagram without chord labels. We interpret this number as the number of linear extensions of the poset determined by the chord diagram of the complete exceptional collection. This also gives an interpretation of complete exceptional sequences as linear extensions.

In Section 6, we give several applications of the theory in type $\mathbb{A}$, including combinatorial proofs that two reddening sequences produce isomorphic ice quivers (see [Kel12] for a general proof in all types using deep category-theoretic techniques) and that there is a bijection between exceptional sequences and certain chains in the lattice of noncrossing partitions.

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2. PRELIMINARIES

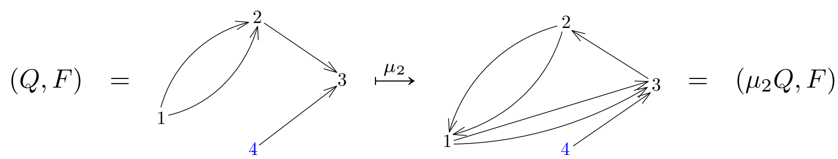
In this section, we recall some basic definitions. We will be interested in the connection of exceptional sequences and the **c**-matrices of an acyclic quiver Q so we begin by defining these. After that we review the basic terminology of quiver representations and exceptional sequences. which serve as the starting point in our study of exceptional sequences. We conclude this section by explaining the notation we will use to discuss exceptional representations of quivers that are orientations of a type $\mathbb{A}_n$ Dynkin diagram.

2.1. **Quiver mutation.** A **quiver** Q is a directed graph without loops or 2-cycles. In other words, Q is a 4-tuple (Q_0, Q_1, s, t), where $Q_0 = [m] := \{1, 2, \ldots, m\}$ is a set of **vertices**, Q_1 is a set of **arrows**, and two functions $s, t : Q_1 \to Q_0$ defined so that for every $a \in Q_1$, we have $s(a) \xrightarrow{a} t(a)$. An **ice quiver** is a pair (Q, F) with Q a quiver and $F \subset Q_0$ **frozen vertices** with the additional restriction that any $i, j \in F$ have no arrows of Q connecting them. We refer to the elements of $Q_0 \backslash F$ as **mutable vertices**. By convention, we assume $Q_0 \backslash F = [n]$ and $F = [n+1, m] := \{n+1, n+2, \ldots, m\}$. Any quiver Q can be regarded as an ice quiver by setting $Q = (Q, \varnothing)$.

The **mutation** of an ice quiver (Q, F) at mutable vertex k, denoted μ_k, produces a new ice quiver $(\mu_k Q, F)$ by the three step process:

(1) For every 2-path $i \to k \to j$ in Q, adjoin a new arrow $i \to j$.
(2) Reverse the direction of all arrows incident to k in Q.
(3) Delete any 2-cycles created during the first two steps.

We show an example of mutation below depicting the mutable (resp. frozen) vertices in black (resp. blue).



The information of an ice quiver can be equivalently described by its (skew-symmetric) **exchange matrix**. Given (Q, F), we define $B = B_{(Q,F)} = (b_{ij}) \in \mathbb{Z}^{n \times m} := \{n \times m \text{ integer matrices}\}$ by $b_{ij} := \#\{i \xrightarrow{a} j \in Q_1\} - \#\{j \xrightarrow{a} i \in Q_1\}$. Furthermore, ice quiver mutation can equivalently be defined as **matrix mutation** of the corresponding exchange matrix. Given an exchange matrix $B \in \mathbb{Z}^{n \times m}$, the **mutation** of B at $k \in [n]$, also denoted μ_k, produces a new exchange matrix $\mu_k(B) = (b'_{ij})$ with entries

$$b'_{ij} := \begin{cases} -b_{ij} & : \text{ if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & : \text{ otherwise.} \end{cases}$$

For example, the mutation of the ice quiver above (here $m = 4$ and $n = 3$) translates into the following matrix mutation. Note that mutation of matrices (or of ice quivers) is an involution (i.e. $\mu_k \mu_k(B) = B$).

$$B_{(Q,F)} = \left[\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{array}\right] \xmapsto{\mu_2} \left[\begin{array}{ccc|c} 0 & -2 & 2 & 0 \\ 2 & 0 & -1 & 0 \\ -2 & 1 & 0 & -1 \end{array}\right] = B_{(\mu_2 Q, F)}.$$

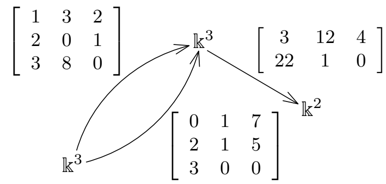
Given a quiver Q, we define its **framed** (resp. **coframed**) quiver to be the ice quiver $\widehat{Q}$ (resp. $\check{Q}$) where $\widehat{Q}_0$ $(= \check{Q}_0) := Q_0 \sqcup [n+1, 2n]$, $F = [n+1, 2n]$, and $\widehat{Q}_1 := Q_1 \sqcup \{i \to n+i : i \in [n]\}$ (resp. $\check{Q}_1 := Q_1 \sqcup \{n+i \to i : i \in [n]\}$). Now given $\widehat{Q}$ we define the **exchange tree** of $\widehat{Q}$, denoted $ET(\widehat{Q})$, to be the (a priori infinite) graph whose vertices are quivers obtained from $\widehat{Q}$ by a finite sequence of mutations and with two vertices connected by an edge if and only if the corresponding quivers are obtained from each other by a single mutation. Similarly, define the **exchange graph** of $\widehat{Q}$, denoted $EG(\widehat{Q})$, to be the quotient of $ET(\widehat{Q})$ where two vertices are identified if and only if there is a **frozen isomorphism** of the corresponding quivers (i.e. an isomorphism that fixes the frozen vertices). Such an isomorphism is equivalent to a simultaneous permutation of the rows and columns of the corresponding exchange matrices.

Given $\widehat{Q}$, we define the **c-matrix** $C(n) = C_R(n)$ (resp. $C = C_R$) of $R \in ET(\widehat{Q})$ (resp. $R \in EG(\widehat{Q})$) to be the submatrix of B_R where $C(n) := (b_{ij})_{i \in [n], j \in [n+1, 2n]}$ (resp. $C := (b_{ij})_{i \in [n], j \in [n+1, 2n]}$). We let **c**-mat$(Q)$ $:= \{C_R : R \in EG(\widehat{Q})\}$. By definition, B_R (resp. C) is only defined up to simultaneous permutations of its rows and first n columns (resp. up to permutations of its rows) for any $R \in EG(\widehat{Q})$.

A row vector of a **c**-matrix, $\overrightarrow{c}$, is known as a **c-vector**. The celebrated theorem of Derksen, Weyman, and Zelevinsky [DWZ10, Theorem 1.7], known as the sign-coherence of **c**-vectors, states that for any $R \in ET(\widehat{Q})$ and $i \in [n]$ the **c**-vector $\overrightarrow{c_i}$ is a nonzero element of $\mathbb{Z}^n_{\geqslant 0}$ or $\mathbb{Z}^n_{\leqslant 0}$. Thus we say a **c**-vector is either **positive** or **negative**.

2.2. **Representations of quivers.** A **representation** $V = ((V_i)_{i \in Q_0}, (\varphi_a)_{a \in Q_1})$ of a quiver Q is an assignment of a $\mathbb{k}$-vector space V_i to each vertex i and a $\mathbb{k}$-linear map $\varphi_a : V_{s(a)} \to V_{t(a)}$ to each arrow a where $\mathbb{k}$ is a field. The **dimension vector** of V is the vector $\underline{\dim}(V) := (\dim V_i)_{i \in Q_0}$. The **support** of V is the set

$\text{supp}(V) := \{i \in Q_0 : V_i \neq 0\}$. Here is an example of a representation, with $\underline{\dim}(V) = (3, 3, 2)$, of the **mutable part** of the quiver depicted in Section 2.1.



Let $V = ((V_i)_{i\in Q_0}, (\varphi_a)_{a\in Q_1})$ and $W = ((W_i)_{i\in Q_0}, (\varrho_a)_{a\in Q_1})$ be two representations of a quiver Q. A **morphism** $\theta : V \to W$ consists of a collection of linear maps $\theta_i : V_i \to W_i$ that are compatible with each of the linear maps in V and W. That is, for each arrow $a \in Q_1$, we have $\theta_{t(a)} \circ \varphi_a = \varrho_a \circ \theta_{s(a)}$. An **isomorphism** of quiver representations is a morphism $\theta : V \to W$ where θ_i is a $\Bbbk$-vector space isomorphism for all $i \in Q_0$. We define $V \oplus W := ((V_i \oplus W_i)_{i\in Q_0}, (\varphi_a \oplus \varrho_a)_{a\in Q_1})$ to be the **direct sum** of V and W. We say that a nonzero representation V is **indecomposable** if it is not isomorphic to a direct sum of two nonzero representations. Note that representations of quivers along with morphisms between them form an abelian category denoted $\text{rep}_\Bbbk(Q)$, with the indecomposable representations forming a full subcategory called $\text{ind}(\text{rep}_\Bbbk(Q))$.

We remark that representations of Q can equivalently be regarded as modules over the **path algebra** $\Bbbk Q$. As such, one can define $\text{Ext}^s_{\Bbbk Q}(V, W)$ $(s \geqslant 0)$ and $\text{Hom}_{\Bbbk Q}(V, W)$ for any representations V and W and $\text{Hom}_{\Bbbk Q}(V, W)$ is isomorphic to the vector space of all morphisms $\theta : V \to W$. We refer the reader to [ASS06] for more details on representations of quivers.

An **exceptional sequence** $\xi = (V_1, \ldots, V_k)$ $(k \leqslant n := \#Q_0)$ is an ordered *list* of **exceptional representations** V_j of Q (i.e. V_j is indecomposable and $\text{Ext}^s_{\Bbbk Q}(V_j, V_j) = 0$ for all $s \geqslant 1$) satisfying $\text{Hom}_{\Bbbk Q}(V_j, V_i) = 0$ and $\text{Ext}^s_{\Bbbk Q}(V_j, V_i) = 0$ if $i < j$ for all $s \geqslant 1$. We define an **exceptional collection** $\overline{\xi} = \{V_1, \ldots, V_k\}$ to be a *set* of exceptional representations V_j of Q that can be ordered in such a way that they define an exceptional sequence. When $k = n$, we say ξ (resp. $\overline{\xi}$) is a **complete exceptional sequence** (CES) (resp. **complete exceptional collection** (CEC)). For Dynkin quivers, a representation is exceptional if and only if it is indecomposable.

The following result of Speyer and Thomas gives a beautiful connection between **c**-matrices of an acyclic quiver Q and CESs. It serves as motivation for our work. Before stating it we remark that for any $R \in ET(\widehat{Q})$ and any $i \in [n]$ where Q is an acyclic quiver, the **c**-vector $\overrightarrow{c_i} = \overrightarrow{c_i}(R) = \pm\underline{\dim}(V_i)$ for some exceptional representation of Q (see [Cha12]). In general, not all indecomposable representations are exceptional. The **c**-vectors are exactly the dimension vectors of the exceptional modules and their negatives.

Notation 2.1. Let $\overrightarrow{c}$ be a **c**-vector of an acyclic quiver Q. Define

$$|\overrightarrow{c}| \quad := \quad \begin{cases} \overrightarrow{c} & : \text{ if } \overrightarrow{c} \text{ is positive} \\ -\overrightarrow{c} & : \text{ if } \overrightarrow{c} \text{ is negative.} \end{cases}$$

Theorem 2.2 ([ST13])**.** Let $C \in \mathbf{c}\text{-mat}(Q)$, let $\{\overrightarrow{c_i}\}_{i\in[n]}$ denote the **c**-vectors of C, and let $|\overrightarrow{c_i}| = \underline{\dim}(V_i)$ for some indecomposable representation of Q. There exists a permutation $\sigma \in \mathfrak{S}_n$ such that $(V_{\sigma(1)}, ..., V_{\sigma(n)})$ is a CES with the property that if there exist positive **c**-vectors in C, then there exists $k \in [n]$ such that $\overrightarrow{c_{\sigma(i)}}$ is positive if and only if $i \in [k, n]$, and $\text{Hom}_{\Bbbk Q}(V_i, V_j) = 0$ if $\overrightarrow{c_i}, \overrightarrow{c_j}$ have the same sign. Conversely, any set of n vectors having these properties defines a **c**-matrix whose rows are $\{\overrightarrow{c_i}\}_{i\in[n]}$.

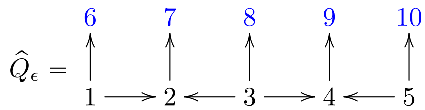
2.3. **Quivers of type $\mathbb{A}_n$.** For the purposes of this paper, we will only be concerned with quivers of type $\mathbb{A}_n$. We say a quiver Q is of **type** $\mathbb{A}_n$ if the underlying graph of Q is a Dynkin diagram of type $\mathbb{A}_n$. By convention, two vertices i and j with $i < j$ in a type $\mathbb{A}_n$ quiver Q are connected by an arrow if and only if $j = i + 1$ and $i \in [n-1]$.

It will be convenient to denote a given type $\mathbb{A}_n$ quiver Q using the notation Q_ϵ, which we now define. Let $\epsilon = (\epsilon_0, \epsilon_1, \ldots, \epsilon_n) \in \{+, -\}^{n+1}$ and for $i \in [n-1]$ define $a_i^{\epsilon_i} \in Q_1$ by

$$a_i^{\epsilon_i} \quad := \quad \begin{cases} i \leftarrow i+1 & : \quad \epsilon_i = - \\ i \rightarrow i+1 & : \quad \epsilon_i = +. \end{cases}$$

Then $Q_\epsilon := ((Q_\epsilon)_0 := [n], (Q_\epsilon)_1 := \{a_i^{\epsilon_i}\}_{i\in[n-1]}) = Q$. One observes that the values of ϵ_0 and ϵ_n do not affect Q_ϵ.

Example 2.3. Let $n = 5$ and $\epsilon = (-, +, -, +, -, +)$ so that $Q_\epsilon = 1 \xrightarrow{a_1^+} 2 \xleftarrow{a_2^-} 3 \xrightarrow{a_3^+} 4 \xleftarrow{a_4^-} 5$. Below we show its framed quiver $\widehat{Q}_\epsilon$.



Let Q_ϵ be given where $\epsilon = (\epsilon_0, \epsilon_1, \ldots, \epsilon_n) \in \{+,-\}^{n+1}$. Let $i, j \in [0, n] := \{0, 1, \ldots, n\}$ where $i < j$ and let $X^\epsilon_{i,j} = ((V_\ell)_{\ell \in (Q_\epsilon)_0}, (\varphi^{i,j}_a)_{a \in (Q_\epsilon)_1}) \in \text{rep}_\Bbbk(Q_\epsilon)$ be the indecomposable representation defined by

$$V_\ell \quad := \quad \begin{cases} \Bbbk & : \quad i+1 \leqslant \ell \leqslant j \\ 0 & : \quad \text{otherwise} \end{cases} \qquad\qquad \varphi^{i,j}_a \quad := \quad \begin{cases} 1 & : \quad a = a_k^{\epsilon_k} \text{ where } i+1 \leqslant k \leqslant j-1 \\ 0 & : \quad \text{otherwise.} \end{cases}$$

The objects of $\text{ind}(\text{rep}_\Bbbk(Q_\epsilon))$ are those of the form $X^\epsilon_{i,j}$ where $0 \leqslant i < j \leqslant n$, up to isomorphism.

3. Strand diagrams

In this section, we define three different types of combinatorial objects: strand diagrams, labeled strand diagrams, and oriented strand diagrams. We will use these objects to classify exceptional collections, exceptional sequences, and **c**-matrices of a given type $\mathbb{A}_n$ quiver Q_ϵ. Throughout this section, we work with a given type $\mathbb{A}_n$ quiver Q_ϵ.

3.1. Exceptional sequences and strand diagrams.

Let $\mathcal{S}_{n,\epsilon} \subset \mathbb{R}^2$ be a collection of $n+1$ points arranged in a horizontal line. We identify these points with $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ where ϵ_j appears to the right of ϵ_i for any $i, j \in [0, n] := \{0, 1, 2, \ldots, n\}$ where $i < j$. Using this identification, we can write $\epsilon_i = (x_i, y_i) \in \mathbb{R}^2$.

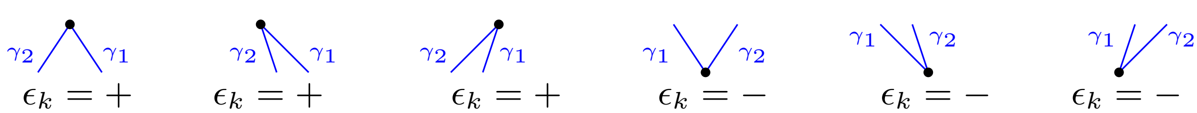
Definition 3.1. Let $i, j \in [0, n]$ where $i \neq j$. A **strand** $c(i, j)$ on $\mathcal{S}_{n,\epsilon}$ is an isotopy class of simple curves in $\mathbb{R}^2$ where any $\gamma \in c(i, j)$ satisfies:

a) the endpoints of γ are ϵ_i and ϵ_j,
b) as a subset of $\mathbb{R}^2$, $\gamma \subset \{(x, y) \in \mathbb{R}^2 : x_i \leqslant x \leqslant x_j\} \backslash \{\epsilon_{i+1}, \epsilon_{i+2}, \ldots, \epsilon_{j-1}\}$,
c) if $k \in [0, n]$ satisfies $i \leqslant k \leqslant j$ and $\epsilon_k = +$ (resp. $\epsilon_k = -$), then γ is locally below (resp. above) ϵ_k.

There is a natural map Φ_ϵ from $\text{ind}(\text{rep}_\Bbbk(Q_\epsilon))$ to the set of strands on $\mathcal{S}_{n,\epsilon}$ given by $\Phi_\epsilon(X^\epsilon_{i,j}) := c(i, j)$.

Remark 3.2. It is clear that any strand can be represented by a monotone curve $\gamma \in c(i, j)$ (i.e. if $t, s \in [0, 1]$ and $t < s$, then $\gamma^{(1)}(t) < \gamma^{(1)}(s)$ where $\gamma^{(1)}$ denotes the **x-coordinate function** of γ.).

We say that two strands $c(i_1, j_1)$ and $c(i_2, j_2)$ **intersect nontrivially** if any two curves $\gamma_\ell \in c(i_\ell, j_\ell)$ with $\ell \in \{1, 2\}$ have at least one transversal crossing. Otherwise, we say that $c(i_1, j_1)$ and $c(i_2, j_2)$ **do not intersect nontrivially**. For example, $c(1, 3), c(2, 4)$ intersect nontrivially if and only if $\epsilon_2 = \epsilon_3$. Additionally, we say that $c(i_2, j_2)$ is **clockwise** from $c(i_1, j_1)$ (or, equivalently, $c(i_1, j_1)$ is **counterclockwise** from $c(i_2, j_2)$) if and only if any $\gamma_1 \in c(i_1, j_1)$ and $\gamma_2 \in c(i_2, j_2)$ share an endpoint ϵ_k and locally appear in one of the following six configurations up to isotopy.



Definition 3.3. A **strand diagram** $d = \{c(i_\ell, j_\ell)\}_{\ell \in [k]}$ ($k \leqslant n$) on $\mathcal{S}_{n,\epsilon}$ is a collection of strands on $\mathcal{S}_{n,\epsilon}$ that satisfies the following conditions:

a) distinct strands do not intersect nontrivially, and
b) the graph determined by d is a **forest** (i.e. a disjoint union of trees)

Let $\mathcal{D}_{k,\epsilon}$ denote the set of strand diagrams on $\mathcal{S}_{n,\epsilon}$ with k strands and let $\mathcal{D}_\epsilon$ denote the set of strand diagrams with any positive number of strands. Then

$$\mathcal{D}_\epsilon = \bigsqcup_{k \in [n]} \mathcal{D}_{k,\epsilon}.$$

Example 3.4. Let $n = 4$ and $\epsilon = (-, +, -, +, +)$ so that $Q_\epsilon = 1 \xrightarrow{a_1^+} 2 \xleftarrow{a_2^-} 3 \xrightarrow{a_3^+} 4$. Then we have that $d_1 = \{c(0, 1), c(0, 2), c(2, 3), c(2, 4)\} \in \mathcal{D}_{4,\epsilon}$ and $d_2 = \{c(0, 4), c(1, 3), c(2, 4)\} \in \mathcal{D}_{3,\epsilon}$. We draw these strand diagrams below.

The following technical lemma classifies when two distinct indecomposable representations of Q_ϵ define 0, 1, or 2 exceptional pairs. Its proof appears in Section 3.2.

Lemma 3.5. Let Q_ϵ be given. Fix two distinct indecomposable representations $U, V \in \text{ind}(\text{rep}_{\Bbbk}(Q_\epsilon))$.

- *a)* The strands $\Phi_\epsilon(U)$ and $\Phi_\epsilon(V)$ intersect nontrivially if and only if neither (U, V) nor (V, U) are exceptional pairs.
- *b)* The strand $\Phi_\epsilon(U)$ is clockwise from $\Phi_\epsilon(V)$ if and only if (U, V) is an exceptional pair and (V, U) is not an exceptional pair.
- *c)* The strands $\Phi_\epsilon(U)$ and $\Phi_\epsilon(V)$ do not intersect at any of their endpoints and they do not intersect nontrivially if and only if (U, V) and (V, U) are both exceptional pairs.

Using Lemma 3.5 we obtain our first main result. The following theorem says that the data of an exceptional collection is completely encoded in the strand diagram it defines.

Theorem 3.6. Let $\overline{\mathcal{E}}_\epsilon := \{\text{exceptional collections of } Q_\epsilon\}$. There is a bijection $\overline{\mathcal{E}}_\epsilon \to \mathcal{D}_\epsilon$ defined by

$$\overline{\xi}_\epsilon = \{X^\epsilon_{i_\ell, j_\ell}\}_{\ell\in[k]} \mapsto \{\Phi_\epsilon(X^\epsilon_{i_\ell, j_\ell})\}_{\ell\in[k]}.$$

Proof. Let $\overline{\xi}_\epsilon = \{X^\epsilon_{i_\ell, j_\ell}\}_{\ell\in[k]}$ be an exceptional collection of Q_ϵ. Let ξ_ϵ be an exceptional sequence gotten from $\overline{\xi}_\epsilon$ by reordering its representations. Without loss of generality, assume $\xi_\epsilon = (X^\epsilon_{i_\ell, j_\ell})_{\ell\in[k]}$ is an exceptional sequence. Thus, $(X^\epsilon_{i_\ell, j_\ell}, X^\epsilon_{i_p, j_p})$ is an exceptional pair for all $\ell < p$. Lemma 3.5 a) implies that distinct strands of $\{\Phi_\epsilon(X^\epsilon_{i_\ell, j_\ell})\}_{\ell\in[k]}$ do not intersect nontrivially.

Now we will show that $\{\Phi_\epsilon(X^\epsilon_{i_\ell, j_\ell})\}_{\ell\in[k]}$ has no cycles. Suppose that $\Phi_\epsilon(X^\epsilon_{i_{\ell_1}, j_{\ell_1}}), ..., \Phi_\epsilon(X^\epsilon_{i_{\ell_p}, j_{\ell_p}})$ is a cycle of length $p \leqslant k$ in $\Phi_\epsilon(\xi_\epsilon)$. Then, there exist $\ell_a, \ell_b \in [k]$ with $\ell_b > \ell_a$ such that $\Phi_\epsilon(X^\epsilon_{i_{\ell_b}, j_{\ell_b}})$ is clockwise from $\Phi_\epsilon(X^\epsilon_{i_{\ell_a}, j_{\ell_a}})$. Thus, by Lemma 3.5 b), $(X^\epsilon_{i_{\ell_a}, j_{\ell_a}}, X^\epsilon_{i_{\ell_b}, j_{\ell_b}})$ is not an exceptional pair. This contradicts the fact that $(X^\epsilon_{i_{\ell_1}, j_{\ell_1}}, ..., X^\epsilon_{i_{\ell_p}, j_{\ell_p}})$ is an exceptional sequence. Hence, the graph determined by $\{\Phi_\epsilon(X^\epsilon_{i_\ell, j_\ell})\}_{\ell\in[k]}$ is a tree. We have shown that $\Phi_\epsilon(\overline{\xi}_\epsilon) \in \mathcal{D}_{k,\epsilon}$.

Now let $d = \{c(i_\ell, j_\ell)\}_{\ell\in[k]} \in \mathcal{D}_{k,\epsilon}$. Since $c(i_\ell, j_\ell)$ and $c(i_m, j_m)$ do not intersect nontrivially, it follows that $(\Phi_\epsilon^{-1}(c(i_\ell, j_\ell)), \Phi_\epsilon^{-1}(c(i_m, j_m)))$ or $(\Phi_\epsilon^{-1}(c(i_m, j_m)), \Phi_\epsilon^{-1}(c(i_\ell, j_\ell)))$ is an exceptional pair for every $\ell \neq m$. Notice that there exists $c(i_{\ell_1}, j_{\ell_1}) \in d$ such that $(\Phi_\epsilon^{-1}(c(i_{\ell_1}, j_{\ell_1})), \Phi_\epsilon^{-1}(c(i_\ell, j_\ell)))$ is an exceptional pair for every $c(i_\ell, j_\ell) \in d \backslash \{c(i_{\ell_1}, j_{\ell_1})\}$. This is true because if such $c(i_{\ell_1}, j_{\ell_1})$ did not exist, then d must have a cycle. Set $E_1 = \Phi_\epsilon^{-1}(c(i_{\ell_1}, j_{\ell_1}))$. Now, choose $c(i_{\ell_p}, j_{\ell_p})$ such that $(\Phi_\epsilon^{-1}(c(i_{\ell_p}, j_{\ell_p})), \Phi_\epsilon^{-1}(c(i_\ell, j_\ell)))$ is an exceptional pair for every $c(i_\ell, j_\ell) \in d \backslash \{c(i_{\ell_1}, j_{\ell_1}), ..., c(i_{\ell_p}, j_{\ell_p})\}$ inductively and put $E_p = \Phi_\epsilon^{-1}(c(i_{\ell_p}, j_{\ell_p}))$. By construction, $(E_1, ..., E_k)$ is a complete exceptional sequence, as desired. □

Our next step is to add distinct integer labels to each strand in a given strand diagram d. When these labels have what we call a **good** labeling, these labels will describe exactly the order in which to put the representations corresponding to strands of d so that the resulting sequence of representations is an exceptional sequence.

Definition 3.7. A **labeled diagram** $d(k) = \{(c(i_\ell, j_\ell), s_\ell)\}_{\ell\in[k]}$ on $\mathcal{S}_{n,\epsilon}$ is a strand diagram on $\mathcal{S}_{n,\epsilon}$ whose strands are labeled by integers $s_\ell \in [k]$ bijectively.

Let $\epsilon_i \in \mathcal{S}_{n,\epsilon}$ and let $((c(i, j_1), s_1), \ldots, (c(i, j_r), s_r))$ be the complete list of labeled strands of $d(k)$ that involve ϵ_i and ordered so that strand $c(i, j_k)$ is clockwise from $c(i, j_{k'})$ if $k' < k$. We say the strand labeling of $d(k)$ is **good** if for each point $\epsilon_i \in \mathcal{S}_{n,\epsilon}$ one has $s_1 < \cdots < s_r$. Let $\mathcal{D}_{k,\epsilon}(k)$ denote the set of labeled strand diagrams on $\mathcal{S}_{n,\epsilon}$ with k strands and with good strand labelings.

Example 3.8. Let $n = 4$ and $\epsilon = (-, +, -, +, +)$ so that $Q_\epsilon = 1 \xrightarrow{a_1^+} 2 \xleftarrow{a_2^-} 3 \xrightarrow{a_3^+} 4$. Below we show the labeled diagrams $d_1(4) = \{(c(0, 1), 1), (c(0, 2), 2), (c(2, 3), 3), (c(2, 4), 4)\}$ and $d_2(3) = \{(c(0, 4), 1), (c(2, 4), 2), (c(1, 3), 3)\}$.

We have that $d_1(4) \in \mathcal{D}_{4,\epsilon}(4)$, but $d_2(3) \notin \mathcal{D}_{3,\epsilon}(3)$.

Theorem 3.9. Let $k \in [n]$ and let $\mathcal{E}_\epsilon(k) := \{\text{exceptional sequences of } Q_\epsilon \text{ of length } k\}$. There is a bijection $\widetilde{\Phi}_\epsilon : \mathcal{E}_\epsilon(k) \to \mathcal{D}_{k,\epsilon}(k)$ defined by

$$\xi_\epsilon = (X^\epsilon_{i_\ell, j_\ell})_{\ell \in [k]} \longmapsto \{(c(i_\ell, j_\ell), k+1-\ell)\}_{\ell \in [k]}.$$

Proof. Let $\xi_\epsilon := (V_1, ..., V_k) \in \mathcal{E}_\epsilon(k)$. By Lemma 3.5 a), $\widetilde{\Phi}_\epsilon(\xi_\epsilon)$ has no strands that intersect nontrivially. Let (V_1, V_2) be an exceptional pair appearing in ξ_ϵ with V_i corresponding to strand c_i in $\widetilde{\Phi}_\epsilon(\xi_\epsilon)$ for $i = 1, 2$, where c_1 and c_2 intersect only at one of their endpoints. Note that by the definition of $\widetilde{\Phi}_\epsilon$, the strand label of c_1 is larger than that of c_2. From Lemma 3.5 b), strand c_1 is clockwise from c_2 in $\widetilde{\Phi}_\epsilon(\xi_\epsilon)$. Thus the strand-labeling of $\widetilde{\Phi}_\epsilon(\xi_\epsilon)$ is good, so $\widetilde{\Phi}_\epsilon(\xi_\epsilon) \in \mathcal{D}_{k,\epsilon}(k)$ for any $\xi_\epsilon \in \mathcal{E}_\epsilon(k)$.

Let $\widetilde{\Psi}_\epsilon : \mathcal{D}_{k,\epsilon}(k) \to \mathcal{E}_\epsilon(k)$ be defined by $\{(c(i_\ell, j_\ell), \ell)\}_{\ell \in [k]} \mapsto (X^\epsilon_{i_k, j_k}, X^\epsilon_{i_{k-1}, j_{k-1}}, ..., X^\epsilon_{i_1, j_1})$. We will show that $\widetilde{\Psi}_\epsilon(d(k)) \in \mathcal{E}_\epsilon(k)$ for any $d(k) \in \mathcal{D}_{k,\epsilon}(k)$ and that $\widetilde{\Psi}_\epsilon = \widetilde{\Phi}_\epsilon^{-1}$. Let

$$\widetilde{\Psi}_\epsilon(\{(c(i_\ell, j_\ell), \ell)\}_{\ell \in [k]}) = (X^\epsilon_{i_k, j_k}, X^\epsilon_{i_{k-1}, j_{k-1}}, \ldots, X^\epsilon_{i_1, j_1}).$$

Consider the pair $(X^\epsilon_{i_s, j_s}, X^\epsilon_{i_{s'}, j_{s'}})$ with $s > s'$. We will show that $(X^\epsilon_{i_s, j_s}, X^\epsilon_{i_{s'}, j_{s'}})$ is an exceptional pair and thus conclude that $\widetilde{\Psi}_\epsilon(\{(c(i_\ell, j_\ell), \ell)\}_{\ell \in [k]}) \in \mathcal{E}_\epsilon(k)$ for any $d(k) \in \mathcal{D}_{k,\epsilon}(k)$. Clearly, $c(i_s, j_s)$ and $c(i_{s'}, j_{s'})$ do not intersect nontrivially. If $c(i_s, j_s)$ and $c(i_{s'}, j_{s'})$ do not intersect at one of their endpoints, then by Lemma 3.5 c) $(X^\epsilon_{i_s, j_s}, X^\epsilon_{i_{s'}, j_{s'}})$ is exceptional. Now suppose $c(i_s, j_s)$ and $c(i_{s'}, j_{s'})$ intersect at one of their endpoints. Because the strand-labeling of $\{(c(i_\ell, j_\ell), \ell)\}_{\ell \in [k]}$ is good, $c(i_s, j_s)$ is clockwise from $c(i_{s'}, j_{s'})$. By Lemma 3.5 b), we have that $(X^\epsilon_{i_s, j_s}, X^\epsilon_{i_{s'}, j_{s'}})$ is exceptional.

To see that $\widetilde{\Psi}_\epsilon = \widetilde{\Phi}_\epsilon^{-1}$, observe that

$$\begin{array}{rcl} \widetilde{\Phi}_\epsilon\left(\widetilde{\Psi}_\epsilon(\{(c(i_\ell, j_\ell), \ell)\}_{\ell \in [k]})\right) & = & \widetilde{\Phi}_\epsilon\left((X^\epsilon_{i_k, j_k}, X^\epsilon_{i_{k-1}, j_{k-1}}, \ldots, X^\epsilon_{i_1, j_1})\right) \\ & = & \{(c(i_\ell, j_\ell), k+1-(k+1-\ell))\}_{\ell \in [k]} \\ & = & \{(c(i_\ell, j_\ell), \ell)\}_{\ell \in [k]}. \end{array}$$

Thus $\widetilde{\Phi}_\epsilon \circ \widetilde{\Psi}_\epsilon = 1_{\mathcal{D}_{n,\epsilon}(k)}$. Similarly, one shows that $\widetilde{\Psi}_\epsilon \circ \widetilde{\Phi}_\epsilon = 1_{\mathcal{E}_\epsilon(k)}$. Thus $\widetilde{\Phi}_\epsilon$ is a bijection. □

The last combinatorial objects we discuss in this section are called **oriented diagrams**. These are strand diagrams whose strands have a direction. We will use these to classify **c**-matrices of a given type $\mathbb{A}_n$ quiver Q_ϵ.

Definition 3.10. An **oriented diagram** $\overrightarrow{d} = \{\overrightarrow{c}(i_\ell, j_\ell)\}_{\ell \in [k]}$ on $\mathcal{S}_{n,\epsilon}$ is a strand diagram on $\mathcal{S}_{n,\epsilon}$ whose strands $\overrightarrow{c}(i_\ell, j_\ell)$ are oriented from ϵ_{i_ℓ} to ϵ_{j_ℓ}.

Remark 3.11. When it is clear from the context what the values of n and ϵ are, we will often refer to a strand diagram on $\mathcal{S}_{n,\epsilon}$ simply as a **diagram**. Similarly, we will often refer to labeled diagrams (resp. oriented diagrams) on $\mathcal{S}_{n,\epsilon}$ as **labeled diagrams** (resp. **oriented diagrams**).

We now define a special subset of the oriented diagrams on $\mathcal{S}_{n,\epsilon}$. As we will see, each element in this subset of oriented diagrams, denoted $\overrightarrow{\mathcal{D}}_{n,\epsilon}$, will correspond to a unique **c**-matrix $C \in$ **c**-mat(Q_ϵ) and vice versa. Thus we obtain a diagrammatic classification of **c**-matrices (see Theorem 3.15).

Definition 3.12. Let $\overrightarrow{\mathcal{D}}_{n,\epsilon}$ denote the set of oriented diagrams $\overrightarrow{d} = \{\overrightarrow{c}(i_\ell, j_\ell)\}_{\ell \in [n]}$ on $\mathcal{S}_{n,\epsilon}$ with the property that any oriented subdiagram $\overrightarrow{d}_1$ of $\overrightarrow{d}$ consisting only of oriented strands connected to ϵ_k in $\mathcal{S}_{n,\epsilon}$ for some $k \in [0, n]$ is a subdiagram of one of the following:

i) $\{\overrightarrow{c}(k, i_1), \overrightarrow{c}(k, i_2), \overrightarrow{c}(j, k)\}$ where $i_1 < k < i_2$ and $\epsilon_k = +$ (shown in Figure 2 (left)),
ii) $\{\overrightarrow{c}(i_1, k), \overrightarrow{c}(i_2, k), \overrightarrow{c}(k, j)\}$ where $i_1 < k < i_2$ and $\epsilon_k = -$ (shown in Figure 2 (right)).

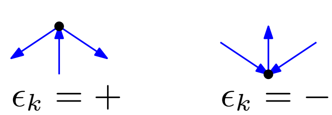


FIGURE 2

Lemma 3.13. Let $\{\overrightarrow{c_i}\}_{i\in[k]}$ be a collection of k **c**-vectors of Q_ϵ where $k \leqslant n$. Let $\overrightarrow{c_i} = \pm\underline{\dim}(X^\epsilon_{i_1,i_2})$ where the sign is determined by $\overrightarrow{c_i}$. If $\{\overrightarrow{c_i}\}_{i\in[k]}$ is a **noncrossing collection** of **c**-vectors (i.e. $\Phi_\epsilon(X^\epsilon_{i_1,i_2})$ and $\Phi_\epsilon(X^\epsilon_{i'_1,i'_2})$ do not intersect nontrivially for any $i, i' \in [k]$), there is an injective map

$$\{\text{noncrossing collections } \{\overrightarrow{c_i}\}_{i\in[k]} \text{ of } Q_\epsilon\} \longrightarrow \{\text{oriented diagrams } \overrightarrow{d} = \{\overrightarrow{c}(i_\ell, j_\ell)\}_{\ell\in[k]}\}$$

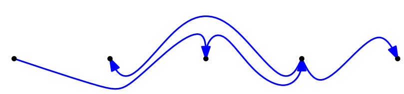
defined by

$$\overrightarrow{c_i} \longmapsto \begin{cases} \overrightarrow{c}(i_1,i_2) & : \overrightarrow{c_i} \text{ is positive} \\ \overrightarrow{c}(i_2,i_1) & : \overrightarrow{c_i} \text{ is negative.} \end{cases}$$

In particular, each **c**-matrix $C_\epsilon \in \mathbf{c}\text{-mat}(Q_\epsilon)$ determines a unique oriented diagram denoted $\overrightarrow{d}_{C_\epsilon}$ with n oriented strands.

Example 3.14. Let $n = 4$ and $\epsilon = (+,+,-,+,-)$ so that $Q_\epsilon = 1 \xrightarrow{a_1^+} 2 \xleftarrow{a_2^-} 3 \xrightarrow{a_3^+} 4$. After performing the mutation sequence $\mu_3 \circ \mu_2$ to the corresponding framed quiver, we have the **c**-matrix with its oriented diagram.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



The following theorem shows oriented diagrams belonging to $\overrightarrow{\mathcal{D}}_{n,\epsilon}$ are in bijection with **c**-matrices of Q_ϵ. We delay its proof until Section 4 because it makes heavy use of the concept of a mixed cobinary tree.

Theorem 3.15. The map $\mathbf{c}\text{-mat}(Q_\epsilon) \to \overrightarrow{\mathcal{D}}_{n,\epsilon}$ induced by the map defined in Lemma 3.13 is a bijection.

3.2. **Proof of Lemma 3.5.** The proof of Lemma 3.5 requires some notions from representation theory of finite dimensional algebras, which we now briefly review. For a more comprehensive treatment of the following notions, we refer the reader to [ASS06].

Definition 3.16. Given a quiver Q with $\#Q_0 = n$, the **Euler characteristic** (of Q) is the $\mathbb{Z}$-bilinear (nonsymmetric) form $\mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ defined by

$$\langle \underline{\dim}(V), \underline{\dim}(W)\rangle = \sum_{i\geqslant 0}(-1)^i \dim \operatorname{Ext}^i_{\Bbbk Q}(V,W)$$

for every $V, W \in \operatorname{rep}_\Bbbk(Q)$.

For hereditary algebras A (e.g. path algebras of acyclic quivers), $\operatorname{Ext}^i_A(V,W) = 0$ for $i \geqslant 2$ and the formula reduces to

$$\langle \underline{\dim}(V), \underline{\dim}(W)\rangle = \dim \operatorname{Hom}_{\Bbbk Q}(V,W) - \dim \operatorname{Ext}^1_{\Bbbk Q}(V,W)$$

The following result gives a simple combinatorial formula for the Euler characteristic. We note that this formula is independent of the orientation of the arrows of Q.

Lemma 3.17. [ASS06, Lemma VII.4.1] Given an acyclic quiver Q with $\#Q_0 = n$ and integral vectors $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in \mathbb{Z}^n$, the Euler characteristic of Q has the form

$$\langle x, y\rangle = \sum_{i\in Q_0} x_i y_i - \sum_{\alpha\in Q_1} x_{s(\alpha)} y_{t(\alpha)}$$

Next, we give a slight simplification of the previous formula. Recall that the support of $V \in \operatorname{rep}_\Bbbk(Q)$ is the set $\operatorname{supp}(V) := \{i \in Q_0 : V_i \neq 0\}$. Thus for quivers of the form Q_ϵ, any representation $X^\epsilon_{i,j} \in \operatorname{ind}(\operatorname{rep}_\Bbbk(Q_\epsilon))$ has $\operatorname{supp}(X^\epsilon_{i,j}) = [i+1, j]$.

Lemma 3.18. Let $X^\epsilon_{k,\ell}, X^\epsilon_{i,j} \in \operatorname{ind}(\operatorname{rep}_\Bbbk(Q_\epsilon))$ and $A := \{a \in (Q_\epsilon)_1 : s(a), t(a) \in \operatorname{supp}(X^\epsilon_{k,\ell}) \cap \operatorname{supp}(X^\epsilon_{i,j})\}$. Then

$$\langle \underline{\dim}(X^\epsilon_{k,\ell}), \underline{\dim}(X^\epsilon_{i,j})\rangle = \chi_{\operatorname{supp}(X^\epsilon_{k,\ell})\cap\operatorname{supp}(X^\epsilon_{i,j})} - \#\left(\{a \in (Q_\epsilon)_1 : s(a) \in \operatorname{supp}(X^\epsilon_{k,\ell}),\ t(a) \in \operatorname{supp}(X^\epsilon_{i,j})\}\backslash A\right)$$

where $\chi_{\operatorname{supp}(X^\epsilon_{k,\ell})\cap\operatorname{supp}(X^\epsilon_{i,j})} = 1$ if $\operatorname{supp}(X^\epsilon_{k,\ell}) \cap \operatorname{supp}(X^\epsilon_{i,j}) \neq \varnothing$ and 0 otherwise.

Proof. We have that

$$\begin{aligned}\langle \underline{\dim}(X^{\epsilon}_{k,\ell}), \underline{\dim}(X^{\epsilon}_{i,j})\rangle &= \sum_{m\in(Q_\epsilon)_0} \underline{\dim}(X^{\epsilon}_{k,\ell})_m \underline{\dim}(X^{\epsilon}_{i,j})_m - \sum_{a\in(Q_\epsilon)_1} \underline{\dim}(X^{\epsilon}_{k,\ell})_{s(a)} \underline{\dim}(X^{\epsilon}_{i,j})_{t(a)} \\ &= \#\left(\operatorname{supp}(X^{\epsilon}_{k,\ell}) \cap \operatorname{supp}(X^{\epsilon}_{i,j})\right) \\ &\quad -\#\{\alpha \in (Q_\epsilon)_1 : s(a) \in \operatorname{supp}(X^{\epsilon}_{k,\ell}),\ t(a) \in \operatorname{supp}(X^{\epsilon}_{i,j})\} \\ &= \#\left(\operatorname{supp}(X^{\epsilon}_{k,\ell}) \cap \operatorname{supp}(X^{\epsilon}_{i,j})\right) - \#A \\ &\quad -\#\left(\{a \in (Q_\epsilon)_1 : s(a) \in \operatorname{supp}(X^{\epsilon}_{k,\ell}),\ t(a) \in \operatorname{supp}(X^{\epsilon}_{i,j})\}\backslash A\right).\end{aligned}$$

Observe that if $\operatorname{supp}(X^{\epsilon}_{k,\ell}) \cap \operatorname{supp}(X^{\epsilon}_{i,j}) \neq \varnothing$, then $\#A = \#(\operatorname{supp}(X^{\epsilon}_{k,\ell}) \cap \operatorname{supp}(X^{\epsilon}_{i,j})) - 1$. Otherwise $\#A = 0$. Thus

$$\langle \underline{\dim}(X^{\epsilon}_{k,\ell}), \underline{\dim}(X^{\epsilon}_{i,j})\rangle = \chi_{\operatorname{supp}(X^{\epsilon}_{k,\ell})\cap \operatorname{supp}(X^{\epsilon}_{i,j})} - \#\left(\{a \in (Q_\epsilon)_1 : s(a) \in \operatorname{supp}(X^{\epsilon}_{k,\ell}),\ t(a) \in \operatorname{supp}(X^{\epsilon}_{i,j})\}\backslash A\right).$$

□

In the sequel, we will use this formula for the Euler characteristic without further comment. We now present several lemmas that will be useful in the proof of Lemma 3.5. The proofs of the next four lemmas use very similar techniques so we only prove Lemma 3.19. The following four lemmas characterize when $\operatorname{Hom}_{\Bbbk Q_\epsilon}(-,-)$ and $\operatorname{Ext}^1_{\Bbbk Q_\epsilon}(-,-)$ vanish for a given type $\mathbb{A}_n$ quiver Q_ϵ. The conditions describing when $\operatorname{Hom}_{\Bbbk Q_\epsilon}(-,-)$ and $\operatorname{Ext}^1_{\Bbbk Q_\epsilon}(-,-)$ vanish are given in terms of inequalities satisfied by the indices that describe a pair of indecomposable representations of Q_ϵ and the entries of ϵ.

Lemma 3.19. Let $X^{\epsilon}_{k,\ell}, X^{\epsilon}_{i,j} \in \operatorname{ind}(\operatorname{rep}_\Bbbk(Q_\epsilon))$. Assume $0 \leqslant i < k < j < \ell \leqslant n$.

i) $\operatorname{Hom}_{\Bbbk Q_\epsilon}(X^{\epsilon}_{i,j}, X^{\epsilon}_{k,\ell}) \neq 0$ if and only if $\epsilon_k = -$ and $\epsilon_j = -$.
ii) $\operatorname{Hom}_{\Bbbk Q_\epsilon}(X^{\epsilon}_{k,\ell}, X^{\epsilon}_{i,j}) \neq 0$ if and only if $\epsilon_k = +$ and $\epsilon_j = +$.
iii) $\operatorname{Ext}^1_{\Bbbk Q_\epsilon}(X^{\epsilon}_{i,j}, X^{\epsilon}_{k,\ell}) \neq 0$ if and only if $\epsilon_k = +$ and $\epsilon_j = +$.
iv) $\operatorname{Ext}^1_{\Bbbk Q_\epsilon}(X^{\epsilon}_{k,\ell}, X^{\epsilon}_{i,j}) \neq 0$ if and only if $\epsilon_k = -$ and $\epsilon_j = -$.

Proof. We only prove *i)* and *iv)* as the proofs of *ii)* is very similar to that of *i)* the proof of *iii)* is very similar to that of *iv)*. To prove *i)*, first assume there is a nonzero morphism $\theta : X^{\epsilon}_{i,j} \to X^{\epsilon}_{k,\ell}$. Clearly, $\theta_s = 0$ if $s \notin [k+1, j]$. If $\theta_s \neq 0$ for some $s \in [n]$, then $\theta_s = \lambda$ for some $\lambda \in \Bbbk^*$ (i.e. θ_s is a nonzero scalar transformation). As θ is a morphism of representations, it must satisfy that for any $a \in (Q_\epsilon)_1$ the equality $\theta_{t(a)}\varphi^{i,j}_a = \varphi^{k,\ell}_a \theta_{s(a)}$ holds. Thus for any $a \in \{a^{\epsilon_{k+1}}_{k+1}, \ldots, a^{\epsilon_{j-1}}_{j-1}\}$, we have $\theta_{t(a)} = \theta_{s(a)}$. As θ is nonzero, this implies that $\theta_s = \lambda$ for any $s \in [k+1, j]$. If $a = a^{\epsilon_k}_k$, then we have

$$\begin{aligned}\theta_{t(a)}\varphi^{i,j}_a &= \varphi^{k,\ell}_a \theta_{s(a)} \\ \theta_{t(a)} &= 0.\end{aligned}$$

Thus $\epsilon_k = -$. Similarly, $\epsilon_j = -$.

Conversely, it is easy to see that if $\epsilon_k = \epsilon_j = -$, then $\theta : X^{\epsilon}_{i,j} \to X^{\epsilon}_{k,\ell}$ defined by $\theta_s = 0$ if $s \notin [k+1, j]$ and $\theta_s = 1$ otherwise is a nonzero morphism.

Next, we prove *iv)*. Observe that by Lemma 3.18 we have

$$\begin{aligned}\dim \operatorname{Ext}^1_{\Bbbk Q_\epsilon}(X^{\epsilon}_{k,\ell}, X^{\epsilon}_{i,j}) &= \dim \operatorname{Hom}_{\Bbbk Q_\epsilon}(X^{\epsilon}_{k,\ell}, X^{\epsilon}_{i,j}) - \langle \underline{\dim}(X^{\epsilon}_{k,\ell}), \underline{\dim}(X^{\epsilon}_{i,j})\rangle \\ &= \dim \operatorname{Hom}_{\Bbbk Q_\epsilon}(X^{\epsilon}_{k,\ell}, X^{\epsilon}_{i,j}) - 1 \\ &\quad +\#\left(\{b \in (Q_\epsilon)_1 : s(b) \in \operatorname{supp}(X^{\epsilon}_{k,\ell}),\ t(b) \in \operatorname{supp}(X^{\epsilon}_{i,j})\}\backslash A\right).\end{aligned}$$

Note that $\#\left(\{b \in (Q_\epsilon)_1 : s(b) \in \operatorname{supp}(X^{\epsilon}_{k,\ell}),\ t(b) \in \operatorname{supp}(X^{\epsilon}_{i,j})\}\backslash A\right) \leqslant 2$. Furthermore, the argument in the first paragraph of the proof shows that $\dim \operatorname{Hom}_{\Bbbk Q_\epsilon}(X^{\epsilon}_{k,\ell}, X^{\epsilon}_{i,j}) \leqslant 1$. By *ii)*, we have that $\operatorname{Ext}^1_{\Bbbk Q_\epsilon}(X^{\epsilon}_{k,\ell}, X^{\epsilon}_{i,j}) \neq 0$ if and only if $\epsilon_k = \epsilon_j = -$. □

Lemma 3.20. Let $X^{\epsilon}_{k,\ell}, X^{\epsilon}_{i,j} \in \operatorname{ind}(\operatorname{rep}_\Bbbk(Q_\epsilon))$. Assume $0 \leqslant i < k < \ell < j \leqslant n$.

i) $\operatorname{Hom}_{\Bbbk Q_\epsilon}(X^{\epsilon}_{i,j}, X^{\epsilon}_{k,\ell}) \neq 0$ if and only if $\epsilon_k = -$ and $\epsilon_\ell = +$.
ii) $\operatorname{Hom}_{\Bbbk Q_\epsilon}(X^{\epsilon}_{k,\ell}, X^{\epsilon}_{i,j}) \neq 0$ if and only if $\epsilon_k = +$ and $\epsilon_\ell = -$.
iii) $\operatorname{Ext}^1_{\Bbbk Q_\epsilon}(X^{\epsilon}_{i,j}, X^{\epsilon}_{k,\ell}) \neq 0$ if and only if $\epsilon_k = +$ and $\epsilon_\ell = -$.
iv) $\operatorname{Ext}^1_{\Bbbk Q_\epsilon}(X^{\epsilon}_{k,\ell}, X^{\epsilon}_{i,j}) \neq 0$ if and only if $\epsilon_k = -$ and $\epsilon_\ell = +$.

Lemma 3.21. Assume $0 \leqslant i < k < j \leqslant n$. Then

i) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{k,j}) = 0$ and $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,j}, X^\epsilon_{i,k}) = 0$.
ii) $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{k,j}) \neq 0$ if and only if $\epsilon_k = +$.
iii) $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,j}, X^\epsilon_{i,k}) \neq 0$ if and only if $\epsilon_k = -$.
iv) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{i,j}) \neq 0$ if and only if $\epsilon_k = -$.
v) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{i,k}) \neq 0$ if and only if $\epsilon_k = +$.
vi) $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{i,j}) = 0$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{i,k}) = 0$.
vii) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,j}, X^\epsilon_{i,j}) \neq 0$ if and only if $\epsilon_k = +$.
viii) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,j}) \neq 0$ if and only if $\epsilon_k = -$.
ix) $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,j}, X^\epsilon_{i,j}) = 0$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,j}) = 0$.

Lemma 3.22. Let $X^\epsilon_{k,\ell}, X^\epsilon_{i,j} \in \mathrm{ind}(\mathrm{rep}_\Bbbk(Q_\epsilon))$. Assume $0 \leqslant i < j < k < \ell \leqslant n$. Then

i) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) = 0$, $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0$,
ii) $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) = 0$, $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0$.

Next, we present three geometric facts about pairs of distinct strands. These geometric facts will be crucial in our proof of Lemma 3.5.

Lemma 3.23. If two distinct strands $c(i_1, j_1)$ and $c(i_2, j_2)$ on $\mathcal{S}_{n,\epsilon}$ intersect nontrivially, then $c(i_1, j_1)$ and $c(i_2, j_2)$ can be represented by a pair of monotone curves that have a unique transversal crossing.

Proof. Suppose $c(i_1, j_1)$ and $c(i_2, j_2)$ intersect nontrivially. Without loss of generality, we assume $i_1 \leqslant i_2$. Let $\gamma_k \in c(i_k, j_k)$ with $k \in [2]$ be monotone curves. There are two cases:

a) $i_1 \leqslant i_2 < j_1 \leqslant j_2$
b) $i_1 \leqslant i_2 < j_2 \leqslant j_1$.

Suppose that case *a)* holds. Let $(x', y') \in \{(x, y) \in \mathbb{R}^2 : x_{i_2} \leqslant x \leqslant x_{j_1}\}$ denote a point where γ_1 crosses γ_2 transversally. If $\epsilon_{i_2} = -$ (resp. $\epsilon_{i_2} = +$), isotope γ_1 relative to ϵ_{i_1} and (x', y') in such a way that the monotonocity of γ_1 is preserved and so that γ_1 lies strictly above (resp. strictly below) γ_2 on $\{(x, y) \in \mathbb{R}^2 : x_{i_2} \leqslant x < x'\}$.

Next, if $\epsilon_{j_1} = -$ (resp. $\epsilon_{j_1} = +$), isotope γ_2 relative to (x', y') and ϵ_{j_2} in such a way that the monotonicity of γ_2 is preserved and so that γ_2 lies strictly above (resp. strictly below) γ_1 on $\{(x, y) \in \mathbb{R}^2 : x' < x \leqslant x_{j_1}\}$. This process produces two monotone curves $\gamma_1 \in c(i_1, j_1)$ and $\gamma_2 \in c(i_2, j_2)$ that have a unique transversal crossing. The proof in case *b)* is very similar. □

Lemma 3.24. Let $c(i_1, j_1)$ and $c(i_2, j_2)$ be distinct strands on $\mathcal{S}_{n,\epsilon}$ that intersect nontrivially. Then $c(i_1, j_1)$ and $c(i_2, j_2)$ do not share an endpoint.

Proof. Suppose $c(i_1, j_1)$ and $c(i_2, j_2)$ share an endpoint. Since $c(i_1, j_1)$ and $c(i_2, j_2)$ intersect nontrivially, then there exist curves $\gamma_k \in c(i_k, j_k)$ with $k \in \{1, 2\}$ that have a unique transversal crossing. However, since $c(i_1, j_1)$ and $c(i_2, j_2)$ share an endpoint, γ_1 and γ_2 are isotopic relative to their endpoints to curves with no transversal crossing. This contradicts that $c(i_1, j_1)$ and $c(i_2, j_2)$ share an endpoint. □

Remark 3.25. If $c(i_1, j_1)$ and $c(i_2, j_2)$ are two distinct strands on $\mathcal{S}_{n,\epsilon}$ that do not intersect nontrivially, then $c(i_1, j_1)$ and $c(i_2, j_2)$ can be represented by a pair of monotone curves $\gamma_\ell \in c(i_\ell, j_\ell)$ where $\ell \in [2]$ that are nonintersecting, except possibly at their endpoints.

We now arrive at the proof of Lemma 3.5. The proof is a case by case analysis where the cases are given in terms of inequalities satisfied by the indices that describe a pair of indecomposable representations of Q_ϵ and the entries of ϵ.

Proof of Lemma 3.5 a). Let $X^\epsilon_{i,j} := U$ and $X^\epsilon_{k,\ell} := V$. Assume that the strands $\Phi_\epsilon(X^\epsilon_{i,j})$ and $\Phi_\epsilon(X^\epsilon_{k,\ell})$ intersect nontrivially. By Lemma 3.24, we can assume without loss of generality that either $0 \leqslant i < k < j < \ell \leqslant n$ or $0 \leqslant i < k < \ell < j \leqslant n$. By Lemma 3.23, we can represent $\Phi_\epsilon(X^\epsilon_{i,j})$ and $\Phi_\epsilon(X^\epsilon_{k,\ell})$ by monotone curves $\gamma_{i,j}$ and $\gamma_{k,\ell}$ that have a unique transversal crossing. Furthermore, we can assume that this unique crossing occurs between ϵ_k and ϵ_{k+1}. There are four possible cases:

i) $\epsilon_k = \epsilon_{k+1} = -$,
ii) $\epsilon_k = -$ and $\epsilon_{k+1} = +$,
iii) $\epsilon_k = \epsilon_{k+1} = +$,
iv) $\epsilon_k = +$ and $\epsilon_{k+1} = -$.

We illustrate these cases up to isotopy in Figure 3. We see that in cases *i)* and *ii)* (resp. *iii)* and *iv)*) $\gamma_{k,\ell}$ lies above (resp. below) $\gamma_{i,j}$ inside of $\{(x, y) \in \mathbb{R}^2 : x_{k+1} \leqslant x \leqslant x_{\min\{\ell,j\}}\}$.

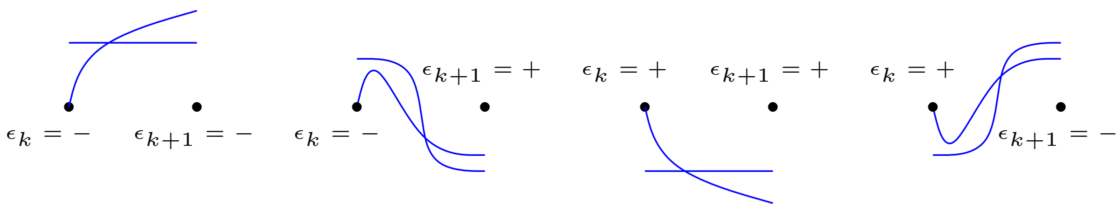


FIGURE 3. The four types of crossings

Suppose $\gamma_{k,\ell}$ lies above $\gamma_{i,j}$ inside $\{(x,y) \in \mathbb{R}^2 : x_{k+1} \leqslant x \leqslant x_{\min\{\ell,j\}}\}$. Then

$$\epsilon_{\min\{\ell,j\}} = \begin{cases} + & : \min\{\ell,j\} = \ell \\ - & : \min\{\ell,j\} = j \end{cases}$$

otherwise $\gamma_{k,\ell}$ and $\gamma_{i,j}$ would have a nonunique transversal crossing. If $\min\{\ell,j\} = \ell$, we have $0 \leqslant i < k < \ell < j \leqslant n$, $\epsilon_k = -$, and $\epsilon_\ell = +$. Now by Lemma 3.20, we have that $\text{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$ and $\text{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) \neq 0$. If $\min\{\ell,j\} = j$, then $0 \leqslant i < k < j < \ell \leqslant n$, $\epsilon_k = -$, and $\epsilon_j = -$. Thus, by Lemma 3.19, we have that $\text{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$ and $\text{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) \neq 0$.

Similarly, if $\gamma_{i,j}$ lies above $\gamma_{k,\ell}$ inside $\{(x,y) \in \mathbb{R}^2 : x_{k+1} \leqslant x \leqslant x_{\min\{\ell,j\}}\}$, it follows that

$$\epsilon_{\min\{\ell,j\}} = \begin{cases} - & : \min\{\ell,j\} = \ell \\ + & : \min\{\ell,j\} = j. \end{cases}$$

If $\min\{\ell,j\} = \ell$, then Lemma 3.20 implies that $\text{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) \neq 0$ and $\text{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$. If $\min\{\ell,j\} = j$, then Lemma 3.19 implies that $\text{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) \neq 0$ and $\text{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$. Thus we conclude that neither $(X^\epsilon_{i,j}, X^\epsilon_{k,\ell})$ nor $(X^\epsilon_{k,\ell}, X^\epsilon_{i,j})$ are exceptional pairs.

Conversely, assume that neither (U,V) nor (V,U) are exceptional pairs where $X^\epsilon_{i,j} := U$ and $X^\epsilon_{k,\ell} := V$. Then at least one of the following is true:

a) $\text{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$ and $\text{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) \neq 0$,
b) $\text{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$ and $\text{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) \neq 0$,
c) $\text{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$ and $\text{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) \neq 0$,
d) $\text{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$ and $\text{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) \neq 0$.

As $X^\epsilon_{i,j}$ and $X^\epsilon_{k,\ell}$ are indecomposable and distinct, we have that $\text{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) = 0$ or $\text{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0$. Without loss of generality, assume that $\text{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0$. Thus *b*) or *d*) hold so $\text{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) \neq 0$. Then Lemma 3.21 and Lemma 3.22 imply that $0 \leqslant i < k < j < \ell \leqslant n$ or $0 \leqslant i < k < \ell < j \leqslant n$.

If $0 \leqslant i < k < j < \ell < n$, we have $\epsilon_k = \epsilon_j = -$ by Lemma 3.19 as $\text{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$ and $\text{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) \neq 0$. Let $\gamma_{i,j} \in \Phi_\epsilon(X^\epsilon_{i,j})$ and $\gamma_{k,\ell} \in \Phi_\epsilon(X^\epsilon_{k,\ell})$. We can assume that there exists $\delta(k) > 0$ such that $\gamma_{i,j}$ and $\gamma_{k,\ell}$ have no transversal crossing inside $\{(x,y) \in \mathbb{R}^2 : x_k \leqslant x \leqslant x_k + \delta(k)\}$. This implies that $\gamma_{i,j}$ lies above $\gamma_{k,\ell}$ inside $\{(x,y) \in \mathbb{R}^2 : x_k \leqslant x \leqslant x_k + \delta(k)\}$. Similarly, we can assume there exists $\delta(j) > 0$ such that $\gamma_{i,j}$ and $\gamma_{k,\ell}$ have no transversal crossing inside $\{(x,y) \in \mathbb{R}^2 : x_j - \delta(j) \leqslant x \leqslant x_j\}$. Thus $\gamma_{i,j}$ lies below $\gamma_{k,\ell}$ inside $\{(x,y) \in \mathbb{R}^2 : x_j - \delta(j) \leqslant x \leqslant x_j\}$. This means $\gamma_{i,j}$ and $\gamma_{k,\ell}$ must have at least one transversal crossing. Thus $\Phi_\epsilon(X^\epsilon_{i,j})$ and $\Phi_\epsilon(X^\epsilon_{k,\ell})$ intersect nontrivially. An analogous argument shows that if $0 \leqslant i < k < \ell < j \leqslant n$, then $\Phi_\epsilon(X^\epsilon_{i,j})$ and $\Phi_\epsilon(X^\epsilon_{k,\ell})$ intersect nontrivially. □

Proof of Lemma 3.5 b). Assume that $\Phi_\epsilon(U)$ is clockwise from $\Phi_\epsilon(V)$. Then we have that one of the following holds:

a) $X^\epsilon_{k,j} = U$ and $X^\epsilon_{i,k} = V$ for some $0 \leqslant i < k < j \leqslant n$,
b) $X^\epsilon_{i,k} = U$ and $X^\epsilon_{k,j} = V$ for some $0 \leqslant i < k < j \leqslant n$,
c) $X^\epsilon_{i,j} = U$ and $X^\epsilon_{i,k} = V$ for some $0 \leqslant i < j \leqslant n$ and $0 \leqslant i < k \leqslant n$,
d) $X^\epsilon_{i,j} = U$ and $X^\epsilon_{k,j} = V$ for some $0 \leqslant i < j \leqslant n$ and $0 \leqslant k < j \leqslant n$.

In Case *a*), we have that $\epsilon_k = -$ since $\Phi_\epsilon(X^\epsilon_{k,j})$ is clockwise from $\Phi_\epsilon(X^\epsilon_{i,k})$. By Lemma 3.21 *i*) and *ii*), we have that $\text{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{k,j}) = 0$ and $\text{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{k,j}) = 0$. Thus $(X^\epsilon_{k,j}, X^\epsilon_{i,k})$ is an exceptional pair. By Lemma 3.21 *iii*), we have that $\text{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,j}, X^\epsilon_{i,k}) \neq 0$. Thus $(X^\epsilon_{i,k}, X^\epsilon_{k,j})$ is not an exceptional pair.

In Case b), we have that $\epsilon_k = +$ since $\Phi_\epsilon(X^\epsilon_{i,k})$ is clockwise from $\Phi_\epsilon(X^\epsilon_{k,j})$. By Lemma 3.21 i) and iii), we have that $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,j}, X^\epsilon_{i,k}) = 0$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,j}, X^\epsilon_{i,k}) = 0$. Thus $(X^\epsilon_{i,k}, X^\epsilon_{k,j})$ is an exceptional pair. By Lemma 3.21 ii), we have that $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{k,j}) \neq 0$. Thus $(X^\epsilon_{k,j}, X^\epsilon_{i,k})$ is not an exceptional pair.

In Case c), if $j < k$, it follows that $\epsilon_j = -$. Indeed, $\Phi_\epsilon(X^\epsilon_{i,j})$ is clockwise from $\Phi_\epsilon(X^\epsilon_{i,k})$ and so by Lemma 3.24 the two do not intersect nontrivially. Now by Remark 3.25, we can choose monotone curves $\gamma_{i,k} \in \Phi_\epsilon(X^\epsilon_{i,k})$ and $\gamma_{i,j} \in \Phi_\epsilon(X^\epsilon_{i,j})$ such that $\gamma_{i,k}$ lies strictly above $\gamma_{i,j}$ on $\{(x,y) \in \mathbb{R}^2 : x_i < x \leqslant x_j\}$. Thus $\epsilon_j = -$. By Lemma 3.21 v) and vi), we have that $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{i,j}) = 0$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{i,j}) = 0$ so that $(X^\epsilon_{i,j}, X^\epsilon_{i,k})$ is an exceptional pair. By Lemma 3.21 iv), we have that $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{i,k}) \neq 0$. Thus $(X^\epsilon_{i,k}, X^\epsilon_{i,j})$ is not an exceptional pair.

Similarly, one shows that if $k < j$, then $\epsilon_k = +$. By Lemma 3.21 iv) and vi), we have that $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{i,j}) = 0$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{i,j}) = 0$ so that $(X^\epsilon_{i,j}, X^\epsilon_{i,k})$ is an exceptional pair. By Lemma 3.21 v), we have that $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{i,k}) \neq 0$. Thus $(X^\epsilon_{i,k}, X^\epsilon_{i,j})$ is not an exceptional pair. The proof in Case d) is completely analogous to the proof in Case c) so we omit it.

Conversely, let $U = X^\epsilon_{i,j}$ and $V = X^\epsilon_{k,\ell}$ and assume that $(X^\epsilon_{i,j}, X^\epsilon_{k,\ell})$ is an exceptional pair and $(X^\epsilon_{k,\ell}, X^\epsilon_{i,j})$ is not an exceptional pair. This implies that at least one of the following holds:

1) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0, \mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0,$ and $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0,$
2) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0, \mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0,$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0.$

By Lemma 3.22, we know that $[i,j] \cap [k,\ell] \neq \varnothing$. This implies that either

i) $\Phi_\epsilon(X^\epsilon_{i,j})$ and $\Phi_\epsilon(X^\epsilon_{k,\ell})$ share an endpoint,
ii) $0 \leqslant i < k < j < \ell \leqslant n$,
iii) $0 \leqslant i < k < \ell < j \leqslant n$,
iv) $0 \leqslant k < i < \ell < j \leqslant n$,
v) $0 \leqslant k < i < j < \ell \leqslant n$.

We will show that $\Phi_\epsilon(X^\epsilon_{i,j})$ and $\Phi_\epsilon(X^\epsilon_{k,\ell})$ share an endpoint.

Suppose $0 \leqslant i < k < j < \ell \leqslant n$. Then since $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0, \mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0$, we have by Lemma 3.19 ii) and iv) that either $\epsilon_k = -$ and $\epsilon_j = +$ or $\epsilon_k = +$ and $\epsilon_j = -$. However, as $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$ or $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$, Lemma 3.19 i) and iii) we have that $\epsilon_k = \epsilon_j = -$ or $\epsilon_k = \epsilon_j = +$. This contradicts that $0 \leqslant i < k < j < \ell \leqslant n$. An analogous argument shows that i, j, k, ℓ do not satisfy $0 \leqslant k < i < \ell < j \leqslant n$.

Suppose $0 \leqslant i < k < \ell < j \leqslant n$. Then since $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0, \mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0$, we have by Lemma 3.20 ii) and iv) that either $\epsilon_k = \epsilon_\ell = +$ or $\epsilon_k = \epsilon_\ell = -$. However, as $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$ or $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$, Lemma 3.20 i) and iii) we have that $\epsilon_k = -$ and $\epsilon_\ell = +$ or $\epsilon_k = +$ and $\epsilon_\ell = -$. This contradicts that $0 \leqslant i < k < \ell < j \leqslant n$. An analogous argmuent shows that i, j, k, ℓ do not satisfy $0 \leqslant k < i < j < \ell \leqslant n$.

We conclude that $\Phi_\epsilon(U)$ and $\Phi_\epsilon(V)$ share an endpoint. Thus we have that one of the following holds where we forget the previous roles played by i, j, k:

a) $X^\epsilon_{k,j} = U$ and $X^\epsilon_{i,k} = V$ for some $0 \leqslant i < k < j \leqslant n$,
b) $X^\epsilon_{i,k} = U$ and $X^\epsilon_{k,j} = V$ for some $0 \leqslant i < k < j \leqslant n$,
c) $X^\epsilon_{i,j} = U$ and $X^\epsilon_{i,k} = V$ for some $0 \leqslant i < j \leqslant n$ and $0 \leqslant i < k \leqslant n$,
d) $X^\epsilon_{i,j} = U$ and $X^\epsilon_{k,j} = V$ for some $0 \leqslant i < j \leqslant n$ and $0 \leqslant k < j \leqslant n$.

Suppose Case a) holds. Then since (U, V) is an exceptional pair, we have $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{k,j}) = 0$. By Lemma 3.21 ii), we have that $\epsilon_k = -$. Thus $\Phi_\epsilon(U)$ is clockwise from $\Phi_\epsilon(V)$.

Suppose Case b) holds. Then since (U, V) is an exceptional pair, we have $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,j}, X^\epsilon_{i,k}) = 0$. By Lemma 3.21 iii), we have that $\epsilon_k = +$. Thus $\Phi_\epsilon(U)$ is clockwise from $\Phi_\epsilon(V)$.

Suppose Case c) holds. Assume $k < j$. Then Lemma 3.21 iv) and the fact that $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{i,j}) = 0$ imply that $\epsilon_k = +$. Thus we have that $\Phi_\epsilon(U) = \Phi_\epsilon(X^\epsilon_{i,j})$ is clockwise from $\Phi_\epsilon(V) = \Phi_\epsilon(X^\epsilon_{i,k})$. Now suppose $j < k$. Then Lemma 3.21 v) and $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{i,j}) = 0$ imply that $\epsilon_j = -$. Thus we have that $\Phi_\epsilon(U) = \Phi_\epsilon(X^\epsilon_{i,j})$ is clockwise from $\Phi_\epsilon(V) = \Phi_\epsilon(X^\epsilon_{i,k})$. The proof in Case d) is very similar so we omit it. □

Proof of Lemma 3.5 c). Observe that two strands $c(i_1, j_1)$ and $c(i_2, j_2)$ share and endpoint if and only if one of the two strands is clockwise from the other. Thus Lemma 3.5 a) and b) implies that $\Phi_\epsilon(U)$ and $\Phi_\epsilon(V)$ do not intersect at any of their endpoints and they do not intersect nontrivially if and only if both (U, V) and (V, U) are exceptional pairs. □

4. Mixed cobinary trees

We recall the definition of an ϵ-mixed cobinary tree and construct a bijection between the set of (isomorphism classes of) such trees and the set of maximal oriented strand diagrams on $\mathcal{S}_{n,\epsilon}$.

Definition 4.1. [IO13] Given a sign function $\epsilon : [0, n] \to \{+, -\}$, an **$\epsilon$-mixed cobinary tree** (MCT) is a tree T embedded in $\mathbb{R}^2$ with vertex set $\{(i, y_i)|i \in [0, n]\}$ and edges straight line segments and satisfying the following conditions.

a) None of the edges is horizontal.
b) If $\epsilon_i = +$ then $y_i \geqslant z$ for any $(i, z) \in T$. So, the tree goes under (i, y_i).
c) If $\epsilon_i = -$ then $y_i \leqslant z$ for any $(i, z) \in T$. So, the tree goes over (i, y_i).
d) If $\epsilon_i = +$ then there is at most one edge descending from (i, y_i) and at most two edges ascending from (i, y_i) and not on the same side.
e) If $\epsilon_i = -$ then there is at most one edge ascending from (i, y_i) and at most two edges descending from (i, y_i) and not on the same side.

Two MCT's T, T' are **isomorphic** as MCT's if there is a graph isomorphism $T \cong T'$ which sends (i, y_i) to (i, y_i') and so that corresponding edges have the same sign of their slopes.

Given a MCT T, there is a partial ordering on $[0, n]$ given by $i <_T j$ if the unique path from (i, y_i) to (j, y_j) in T is monotonically increasing. Isomorphic MCT's give the same partial ordering by definition. Conversely, the partial ordering $<_T$ determines T uniquely up to isomorphism since T is the Hasse diagram of the partial ordering $<_T$. We sometimes decorate MCT's with **leaves** at vertices so that the result is **trivalent**, i.e., with three edges incident to each vertex. See, e.g., Figure 5. The ends of these leaves are not considered to be vertices. In that case, each vertex with $\epsilon = +$ forms a "Y" and this pattern is vertically inverted for $\epsilon = -$. The position of the leaves is uniquely determined.

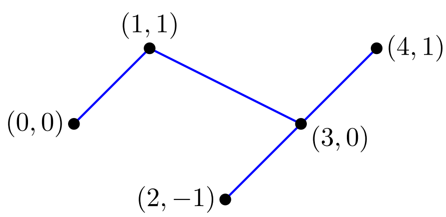


Figure 4. A MCT with $\epsilon_1 = \epsilon_2 = -$, $\epsilon_3 = +$ and any value for ϵ_0, ϵ_4.

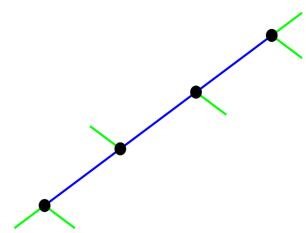


Figure 5. This MCT (in blue) has added green leaves showing that $\epsilon = (-, +, -, -)$.

In Figure 5, the four vertices have coordinates $(0, y_0), (1, y_1), (2, y_2), (3, y_3)$ where y_i can be any real numbers so that $y_0 < y_1 < y_2 < y_3$. This inequality defines an open subset of $\mathbb{R}^4$ which is called the region of this tree T. More generally, for any MCT T, the **region** of T, denoted $\mathcal{R}_\epsilon(T)$, is the set of all points $y \in \mathbb{R}^{n+1}$ with the property that there exists a mixed cobinary tree T' which is isomorphic to T so that the vertex set of T' is $\{(i, y_i) \,|\, i \in [n]\}$.

Theorem 4.2. [IO13] Let n and $\epsilon : [n] \to \{+, -\}$ be fixed. Then, for every MCT T, the region $\mathcal{R}_\epsilon(T)$ is convex and nonempty. Furthermore, every point $y = (y_0, \cdots, y_n)$ in $\mathbb{R}^{n+1}$ with distinct coordinates lies in $\mathcal{R}_\epsilon(T)$ for a unique T (up to isomorphism). In particular these regions are disjoint and their union is dense in $\mathbb{R}^{n+1}$.

For a fixed n and $\epsilon : [n] \to \{+, -\}$ we will construct a bijection between the set $\mathcal{T}_\epsilon$ of isomorphism classes of mixed cobinary trees with sign function ϵ and the set $\overrightarrow{\mathcal{D}}_{n,\epsilon}$ defined in Definition 3.12.

Lemma 4.3. Let $\overrightarrow{d} = \{\overrightarrow{c}(i_\ell, j_\ell)\}_{\ell \in [n]} \in \overrightarrow{\mathcal{D}}_{n,\epsilon}$. Let p, q be two points on this graph so that q lies directly above p. Then each edge of $\overrightarrow{d}$ in the unique path γ from p to q is oriented in the same direction as γ.

Proof. The proof will be by induction on the number m of internal vertices of the path γ. If $m = 1$ with internal vertex ϵ_i then the path γ has only two edges of $\overrightarrow{d}$: one going from p to ϵ_i, say to the left, and the other going from ϵ_i back to q. Since $\overrightarrow{d} \in \overrightarrow{\mathcal{D}}_{n,\epsilon}$, the edge coming into ϵ_i from its right is below the edge going out from ϵ_i to q. Therefore the orientation of these two edges in $\overrightarrow{d}$ matches that of γ.

Now suppose that $m \geqslant 2$ and the lemma holds for smaller m. There are two cases. Case 1: The path γ lies entirely on one side of p and q (as in the case $m = 1$). Case 2: γ has internal vertices on both sides of p, q.

Case 1: Suppose by symmetry that γ lies entirely on the left side of p and q. Let j be maximal so that ϵ_j is an internal vertex of γ. Then γ contains an edge connecting ϵ_j to either p or q, say p. And the edge of γ ending in q contains a unique point r which lies above ϵ_j. This forces the sign to be $\epsilon_j = -$. By induction on m, the rest of the path γ, which goes from ϵ_j to r has orientation compatible with that of $\overrightarrow{d}$. So, it must be oriented outward from ϵ_j. Any other edge at ϵ is oriented inward. So, the edge from p to ϵ_j is oriented from p to ϵ_j as required. The edge coming into r from the left is oriented to the right (by induction). So, this same edge continues to be oriented to the right as it goes from r to q. The other subcases (when ϵ_j is connected to q instead of p and when γ lies to the right of p and q) are analogous.

Case 2: Suppose that γ on both sides of p and q. Then γ passes through a third point, say r, on the same vertical line containing p and q. Let γ_0 and γ_1 denote the parts of γ going from p to r and from r to q respectively. Then γ_0, γ_1 each have at least one internal vertex. So, the lemma holds for each of them separately. There are three subcases: either (a) r lies below p, (b) r lies above q or (c) r lies between p and q. In subcase (a), we have, by induction on m, that γ_0, γ_1 are both oriented away from r. So, $r = \epsilon_k = +$ which contradicts the assumption that q lies above r. Similarly, subcase (c) is not possible. In subcase (b), we have by induction on m that the orientations of the edges of $\overrightarrow{d}$ are compatible with the orientations of γ_0 and γ_1. So, the lemma holds in subcase (b), which is the only subcase of Case 2 which is possible. Therefore, the lemma holds in all cases. □

Theorem 4.4. For each $\overrightarrow{d} = \{\overrightarrow{c}(i_\ell, j_\ell)\}_{\ell \in [n]} \in \overrightarrow{\mathcal{D}}_{n,\epsilon}$, let $\mathcal{R}(\overrightarrow{d})$ denote the set of all $y \in \mathbb{R}^{n+1}$ so that $y_i < y_j$ for any $\overrightarrow{c}(i,j)$ in $\overrightarrow{d}$. Then $\mathcal{R}(\overrightarrow{d}) = \mathcal{R}_\epsilon(T)$ for a uniquely determined mixed cobinary tree T. Furthermore, this gives a bijection

$$\overrightarrow{\mathcal{D}}_{n,\epsilon} \cong \mathcal{T}_\epsilon.$$

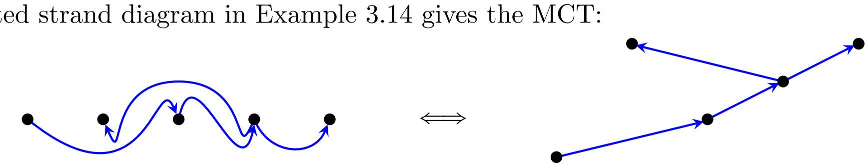
Proof. We first verify the existence of a mixed cobinary tree T for every choice of $y \in \mathcal{R}(\overrightarrow{d})$. Since the strand diagram is a tree, the vector y is uniquely determined by $y_0 \in \mathbb{R}$ and $y_{j_\ell} - y_{i_\ell} > 0$, $\ell \in [n]$, which are arbitrary. Given such a y, we need to verify that the n line segments L_ℓ in $\mathbb{R}^2$ connecting the pairs of points $(i_\ell, y_{i_\ell}), (j_\ell, y_{j_\ell})$ meet only on their endpoints. This follows from the lemma above. If two of these line segments, say L_k, L_ℓ, meet then they come from two distinct points $p \in \overrightarrow{c}(i_k, j_k)$ and $q = \overrightarrow{c}(i_\ell, j_\ell)$ in the strand diagram which lie one the same vertical line. If q lies above p in the strand diagram then, by Lemma 4.3, the unique path γ from p to q is oriented positively. This implies that the y coordinate of the point in L_k corresponding to p is less that the y coordinate of the point in L_ℓ corresponding to q. Thus, this intersection is not possible. So, T is a linearly embedded tree. The lemma also implies that the tree T lies above all negative vertices and below all positive vertices. The other parts of Definition 4.1 follow from the definition of an oriented strand diagram. Therefore $T \in \mathcal{T}_\epsilon$. Since this argument works for every $y \in \mathcal{R}(\overrightarrow{d})$, we see that $\mathcal{R}(\overrightarrow{d}) = \mathcal{R}_\epsilon(T)$ as claimed.

A description of the inverse mapping $\mathcal{T}_\epsilon \to \overrightarrow{\mathcal{D}}_{n,\epsilon}$ is given as follows. Take any MCT T and deform the tree by moving all vertices vertically to the subset $[n] \times 0$ on the x-axis and deforming the edges in such a way that they are always embedded in the plane with no vertical tangents and so that their interiors do not meet. The result is an oriented strand diagram $\overrightarrow{d}$ with $\mathcal{R}(\overrightarrow{d}) = \mathcal{R}_\epsilon(T)$.

It is clear that these are inverse mappings giving the desired bijection $\overrightarrow{\mathcal{D}}_{n,\epsilon} \cong \mathcal{T}_\epsilon$. □

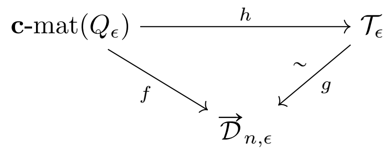
Example 4.5. The MCTs in Figures 4 and 5 above give the oriented strand diagrams:

and the oriented strand diagram in Example 3.14 gives the MCT:



We now arrive at the proof of Theorem 3.15. This theorem follows from the fact that oriented diagrams belonging to $\overrightarrow{\mathcal{D}}_{n,\epsilon}$ can be regarded as mixed cobinary trees by Theorem 4.4.

Proof of Theorem 3.15. Let f be the map $\mathbf{c}\text{-mat}(Q_\epsilon) \to \overrightarrow{\mathcal{D}}_{n,\epsilon}$ induced by the map defined in Lemma 3.13, and let g be the bijective map $\mathcal{T}_\epsilon \to \overrightarrow{\mathcal{D}}_{n,\epsilon}$ defined in Theorem 4.4. We will assert the existence of a map $h : \mathbf{c}\text{-mat}(Q_\epsilon) \to \mathcal{T}_\epsilon$ which fits into the diagram



The theorem will follow after verifying that h is a bijection and that $f = g \circ h$.

We will define two new notions of **c**-matrix, one for MCT's and one for oriented strand diagrams. Let $T \in \mathcal{T}_\epsilon$ with internal edges ℓ_i having endpoints (i_1, y_{i_1}) and (i_2, y_{i_2}). For each ℓ_i, define the '**c**-vector' of ℓ_i to be $c_i(T) := \sum_{i_1 < j \leqslant i_2} \operatorname{sgn}(\ell_i) e_j$, where $\operatorname{sgn}(\ell_i)$ is the sign of the slope of ℓ_i. Define $c(T)$ to be the '**c**-matrix' of T whose rows are the **c**-vectors $c_i(T)$. Now, let $\overrightarrow{d} = \{\overrightarrow{c}(i_\ell, j_\ell)\}_{\ell \in [n]} \in \overrightarrow{\mathcal{D}}_{n,\epsilon}$. For each oriented strand $\overrightarrow{c}(i_\ell, j_\ell)$, define the '**c**-vector' of $\overrightarrow{c}(i_\ell, j_\ell)$ to be

$$c_\ell(\overrightarrow{d}) \quad := \quad \begin{cases} \sum_{i_\ell < k \leqslant j_\ell} \operatorname{sgn}(\overrightarrow{c}(i_\ell, j_\ell)) e_k & : \quad i_\ell < j_\ell \\ \sum_{j_\ell < k \leqslant i_\ell} \operatorname{sgn}(\overrightarrow{c}(i_\ell, j_\ell)) e_k & : \quad i_\ell > j_\ell \end{cases}$$

where $\operatorname{sgn}(\overrightarrow{c}(i_\ell, j_\ell))$ is positive if $i_\ell < j_\ell$ and negative if $i_\ell > j_\ell$. Define $c(\overrightarrow{d})$ to be the '**c**-matrix' of $\overrightarrow{d}$ whose rows are the **c**-vectors $c_\ell(\overrightarrow{d})$.

It is known that the notion of **c**-matrix for MCT's coincides with the original notion of **c**-matrix defined in Section 2.1, and that there is a bijection between $\mathbf{c}\text{-mat}(Q_\epsilon)$ and $\mathcal{T}_\epsilon$ which preserves **c**-matrices (see [IO13, Remarks 2 and 4] for details). Thus, we have a bijective map $h : \mathbf{c}\text{-mat}(Q_\epsilon) \to \mathcal{T}_\epsilon$. On the other hand, the bijection $g : \mathcal{T}_\epsilon \to \overrightarrow{\mathcal{D}}_{n,\epsilon}$ defined in Theorem 4.4 also preserves **c**-matrices. The map $f : \mathbf{c}\text{-mat}(Q_\epsilon) \to \mathcal{T}_\epsilon$ preserves **c**-matrices by definition. Hence, we have $f = g \circ h$ and f is a bijection, as desired. $\square$

Remark 4.6. For linearly-ordered quivers (those with $\epsilon = (+, ..., +)$ or $\epsilon = (-, ..., -)$), this bijection was established by the first and third authors in [GM15] using a different approach. The bijection was given by hand without going through MCT's. This was more tedious, and the authors feel that some aspects (such as mutation) are better phrased in terms of MCT's.

5. Exceptional sequences and linear extensions

In this section, we consider the problem of counting the number of CESs arising from a given CEC. We show that this problem can be restated as the problem of counting the number of linear extensions of certain posets. Throughout this section we fix a strand diagram $d = \{c(i_\ell, j_\ell)\}_{\ell \in [n]}$ on $\mathcal{S}_{n,\epsilon}$.

Definition 5.1. We define the **poset** $\mathcal{P}_d = (\{c(i_\ell, j_\ell)\}_{\ell \in [n]}, \leqslant)$ **associated to** d as the partially ordered set whose elements are the strands of d with covering relations given by $c(i, j) \lessdot c(k, \ell)$ if and only if the strand $c(k, \ell)$ is clockwise from $c(i, j)$ and there does not exist another strand $c(i', j')$ distinct from $c(i, j)$ and $c(k, \ell)$ such that $c(i', j')$ is clockwise from $c(i, j)$ and counterclockwise from $c(k, \ell)$.

The construction defines a poset because any oriented cycle in the Hasse diagram of $\mathcal{P}_d$ arises from a cycle in the underlying graph of d. Since the underlying graph of d is a tree, the diagram d has no cycles. In Figure 6, we show a diagram $d \in \mathcal{D}_{4,\epsilon}$ where $\epsilon := (-, +, -, +, +)$ and its poset $\mathcal{P}_d$.

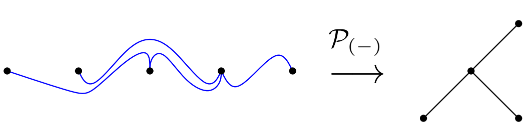


Figure 6. A diagram and its poset.

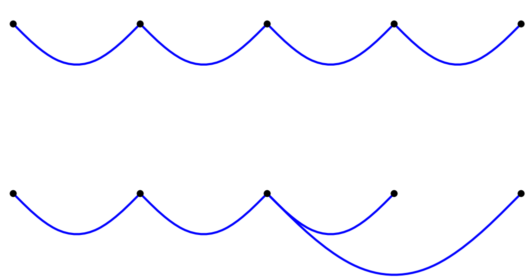
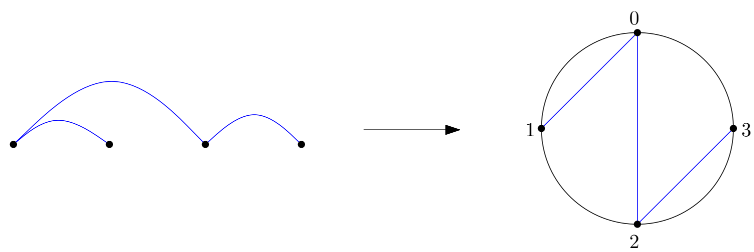


Figure 7. Two diagrams with the same poset.

Let $\mathcal{P}$ be a finite poset with $m = \#\mathcal{P}$. Let $f : \mathcal{P} \to \mathbf{m}$ be an injective, order-preserving map (i.e. $x \leqslant y$ implies $f(x) \leqslant f(y)$ for all $x, y \in \mathcal{P}$) where $\mathbf{m}$ is the linearly-ordered poset with m elements. We call f a **linear extension** of $\mathcal{P}$. We denote the set of linear extensions of $\mathcal{P}$ by $\mathscr{L}(\mathcal{P})$. Note that since f is an injective map between sets of the same cardinality, f is a bijective map between those sets.

In general, the map $\mathcal{D}_{n,\epsilon} \to \mathscr{P}(\mathcal{D}_{n,\epsilon}) := \{\mathcal{P}_d : d \in \mathcal{D}_{n,\epsilon}\}$ is not injective. For instance, each of the two diagrams in Figure 7 have $\mathcal{P}_d = \mathbf{4}$ where $\mathbf{4}$ denotes the linearly-ordered poset with 4 elements. It is thus natural to ask which posets are obtained from strand diagrams.

Our next result describes the posets arising from diagrams in $\mathcal{D}_{n,\epsilon}$ where $\epsilon = (-, \dots, -)$ or $\epsilon = (+, \dots, +)$. Before we state it, we remark that diagrams in $\mathcal{D}_{n,\epsilon}$ where $\epsilon = (-, \dots, -)$ or $\epsilon = (+, \dots, +)$ can be regarded as **chord diagrams**.[1] The following example shows the simple bijection.



Let $d \in \mathcal{D}_{n,\epsilon}$ where $\epsilon = (-, \dots, -)$ or $\epsilon = (+, \dots, +)$. Let $c(i, j)$ be a strand of d. There is an obvious action of $\mathbb{Z}/(n+1)\mathbb{Z}$ on chord diagrams. Let $\tau \in \mathbb{Z}/(n+1)\mathbb{Z}$ denote a generator and define $\tau c(i, j) := c(i-1, j-1)$ and $\tau^{-1}c(i, j) := c(i+1, j+1)$ where we consider $i \pm 1$ and $j \pm 1$ mod $n+1$. We also define $\tau d := \{\tau c(i_\ell, j_\ell)\}_{\ell \in [n]}$ and $\tau^{-1}d := \{\tau^{-1}c(i_\ell, j_\ell)\}_{\ell \in [n]}$. The next lemma, which is easily verified, shows that the order-theoretic properties of CECs are invariant under the action of $\tau^{\pm 1}$.

Lemma 5.2. Let $d \in \mathcal{D}_{n,\epsilon}$ where $\epsilon = (-, \dots, -)$ or $\epsilon = (+, \dots, +)$. Then we have the following isomorphisms of posets $\mathcal{P}_d \cong \mathcal{P}_{\tau d}$ and $\mathcal{P}_d \cong \mathcal{P}_{\tau^{-1}d}$.

Theorem 5.3. Let $\epsilon = (-, \dots, -)$ or let $\epsilon = (+, \dots, +)$. Then a poset $\mathcal{P} \in \mathscr{P}(\mathcal{D}_{n,\epsilon})$ if and only if

i) each $x \in \mathcal{P}$ has at most two covers and covers at most two elements,
ii) the underlying graph of the Hasse diagram of $\mathcal{P}$ has no cycles,
iii) the Hasse diagram of $\mathcal{P}$ is connected.

Proof. Let $\mathcal{P}_d \in \mathscr{P}(\mathcal{D}_{n,\epsilon})$. By definition, $\mathcal{P}_d$ satisfies i). It is also clear that the Hasse diagram of $\mathcal{P}_d$ is connected since d is a connected graph. To see that $\mathcal{P}_d$ satisfies ii), suppose that C is a full subposet of $\mathcal{P}_d$ whose Hasse diagram is a **minimal cycle** (i.e. the underlying graph of C is a cycle, but does not contain a proper subgraph that is a cycle). Thus there exists $x_C \in \mathcal{P}_d$ such that $x_C \in C$ is covered by two distinct elements $y, z \in C$. Observe that C can be regarded as a sequence of chords $\{c_i\}_{i=0}^{\ell}$ of d in which y and z appear exactly once and where for all $i \in [0, \ell]$ c_i and c_{i+1} (we consider the indices modulo $\ell + 1$) share a marked point j and no chord adjacent to j appears between c_i and c_{i+1}. Since the chords of d are noncrossing, such a sequence cannot exist. Thus the Hasse diagram of $\mathcal{P}_d$ has no cycles.

To prove the converse, we proceed by induction on the number of elements of $\mathcal{P}$ where $\mathcal{P}$ is a poset satisfying conditions $i), ii), iii)$. If $\#\mathcal{P} = 1$, then $\mathcal{P}$ is the unique poset with one element and $\mathcal{P} = \mathcal{P}_d$ where d is the unique chord diagram associated to the disk with two marked points that is a spanning tree. Assume that for any poset $\mathcal{P}$ satisfying conditions $i), ii), iii)$ with $\#\mathcal{P} = r$ for any positive integer $r < n+1$ there exists a chord diagram d such that $\mathcal{P} = \mathcal{P}_d$. Let $\mathcal{Q}$ be a poset satisfying the above conditions and assume $\#\mathcal{Q} = n+1$. Let $x \in \mathcal{Q}$ be a maximal element.

Assume x covers two elements $y, z \in \mathcal{Q}$. Then the poset $\mathcal{Q} - \{x\} = \mathcal{Q}_1 + \mathcal{Q}_2$ where $y \in \mathcal{Q}_1$, $z \in \mathcal{Q}_2$, and $\mathcal{Q}_i$ satisfies $i), ii), iii)$ for $i \in [2]$. By induction, there exists positive integers k_1, k_2 satisfying $k_1 + k_2 = n$ and diagrams

$$d_i \in \mathcal{D}_{k_i, \epsilon^{(i)}} := \{\text{diagrams } \{c_i(i_\ell, j_\ell)\}_{\ell \in [k_i]} \text{ in a disk with } k_i + 1 \text{ marked points}\}$$

where $\mathcal{Q}_i = \mathcal{P}_{d_i}$ for $i \in [2]$ and where $\epsilon^{(i)} \in \{+, -\}^{k_i+1}$ has all of its entries equal to the entries of ϵ. By Lemma 5.2, we can further assume that the chord corresponding to $y \in \mathcal{Q}_1$ (resp. $z \in \mathcal{Q}_2$) is $c_1(i(y), k_1) \in d_1$ for

[1]These noncrossing trees embedded in a disk with vertices lying on the boundary have been studied by Araya in [Ara13], Goulden and Yong in [GY02], and the first and third authors in [GM15].

some $i(y) \in [0, k_1 - 1]$ (resp. $c_2(j(z), k_2) \in d_2$ for some $j(z) \in [1, k_2]$). Define $d_1 \sqcup d_2 := \{c'(i'_\ell, j'_\ell)\}_{\ell \in [n]}$ to be the diagram in the disk with $n + 2$ marked points as follows:

$$c'(i'_\ell, j'_\ell) \quad := \quad \begin{cases} c_1(i_\ell, j_\ell) & : \text{ if } \ell \in [k_1] \\ \tau^{-(k_1+1)} c_2(i_{\ell-k_1}, j_{\ell-k_1}) & : \text{ if } \ell \in [k_1 + 1, n]. \end{cases}$$

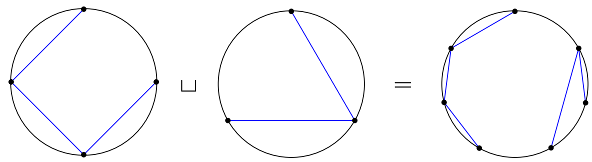


FIGURE 8. An example with $k_1 = 3$ and $k_2 = 2$ so that $n = k_1 + k_2 = 5$.

Define $c'(i'_{n+1}, j'_{n+1}) := c(k_1, n + 1)$ and then $d := \{c'(i'_\ell, j'_\ell)\}_{\ell \in [n+1]}$ satisfies $i), ii), iii)$, and $\mathcal{Q} = \mathcal{P}_d$.

If the Hasse diagram of $\mathcal{Q} - \{x\}$ is connected, then by induction the poset $\mathcal{Q} - \{x\} = \mathcal{P}_d$ for some diagram $d = \{c(i_\ell, j_\ell)\}_{\ell \in [n]} \in \mathcal{D}_{n,\epsilon}$ where we assume $i_\ell < j_\ell$. Since the Hasse diagram of $\mathcal{Q} - \{x\}$ is connected, it follows that x covers a unique element in $\mathcal{Q}$. Let $y = c(i(y), j(y)) \in \mathcal{Q} - \{x\}$ $(i(y) < j(y))$ denote the unique element that is covered by x in $\mathcal{Q}$. This means that there are no chords in d obtained by a clockwise rotation of $c(i(y), j(y))$ about $i(y)$ or there are no chords in d obtained by a clockwise rotation of $c(i(y), j(y))$ about $j(y)$. Without loss of generality, we assume that there are no chords in d obtained by a clockwise rotation of $c(i(y), j(y))$ about $i(y)$.

Regard d as an element of $\mathcal{D}_{n+1,\epsilon'}$ where $\epsilon' \in \{+, -\}^{n+2}$ has all of its entries equal to the entries of ϵ as follows. Replace it with $\widetilde{d} := \{c'(i'_\ell, j'_\ell)\}_{\ell \in [n]} \in \mathcal{D}_{n+1,\epsilon'}$ defined by (we give an example of this operation below with $n = 6$)

$$c'(i'_\ell, j'_\ell) \quad := \quad \begin{cases} \rho^{-1} c(i_\ell, j_\ell) & : \text{ if } i_\ell \leqslant i(y) \text{ and } j(y) \leqslant j_\ell, \\ \tau^{-1} c(i_\ell, j_\ell) & : \text{ if } j(y) \leqslant i_\ell, \\ c(i_\ell, j_\ell) & : \text{ otherwise.} \end{cases}$$

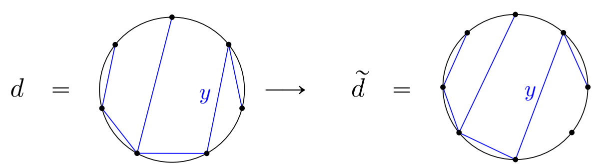


FIGURE 9. An example with $n = 6$.

Define $c'(i'_{n+1}, j'_{n+1}) := c(i(y), i(y) + 1)$ and put $d' := \{c'(i'_\ell, j'_\ell)\}_{\ell \in [n+1]}$. As $\mathcal{Q} - \{x\}$ satisfies $i), ii)$, and $iii)$, it is clear that the resulting chord diagram d' satisfies $\mathcal{P} = \mathcal{P}_{d'}$. □

Theorem 5.4. Let $d = \{c(i_\ell, j_\ell)\}_{\ell \in [n]} \in \mathcal{D}_{n,\epsilon}$ and let $\overline{\xi}_\epsilon$ denote the corresponding complete exceptional collection. Let $\text{CES}(\overline{\xi}_\epsilon)$ denote the set of CESs that can be formed using only the representations appearing in $\overline{\xi}_\epsilon$. Then the map $\chi : \text{CES}(\overline{\xi}_\epsilon) \to \mathscr{L}(\mathcal{P}_d)$ defined by $(X^\epsilon_{i_1,j_1}, \ldots, X^\epsilon_{i_n,j_n}) \overset{\chi_2}{\longmapsto} \{(c(i_\ell, j_\ell), n + 1 - \ell)\}_{\ell \in [n]} \overset{\chi_1}{\longmapsto} (f(c(i_\ell, j_\ell)) := n + 1 - \ell)$ is a bijection.

Proof. The map $\chi_2 = \Phi : \text{CES}(\overline{\xi}_\epsilon) \to \mathcal{D}_{n,\epsilon}(n)$ is a bijection by Theorem 3.9. Thus it is enough to prove that $\chi_1 : \mathcal{D}_{n,\epsilon}(n) \to \mathscr{L}(\mathcal{P}_d)$ is a bijection.

First, we show that $\chi_1(d(n)) \in \mathscr{L}(\mathcal{P}_d)$ for any $d(n) \in \mathcal{D}_{n,\epsilon}(n)$. Let $d(n) \in \mathcal{D}_{n,\epsilon}(n)$ and let $f := \chi_1(d(n))$. Since the strand-labeling of $d(n)$ is good, if (c_1, ℓ_1) and (c_2, ℓ_2) are two labeled strands of $d(n)$ satisfying $c_1 \leqslant c_2$, then $f(c_1) = \ell_1 \leqslant \ell_2 = f(c_2)$. Thus f is order-preserving. As the strands of $d(n)$ are bijectively labeled by $[n]$, we have that f is bijective so $f \in \mathscr{L}(\mathcal{P}_d)$.

Next, define a map

$$\begin{array}{ccc} \mathscr{L}(\mathcal{P}_d) & \overset{\varphi}{\longrightarrow} & \mathcal{D}_{n,\epsilon}(n) \\ f & \longmapsto & \{(c(i_\ell, j_\ell), f(c(i_\ell, j_\ell)))\}_{\ell \in [n]}. \end{array}$$

To see that $\varphi(f) \in \mathcal{D}_{n,\epsilon}(n)$ for any $f \in \mathscr{L}(\mathcal{P}_d)$, consider two labeled strands $(c_1, f(c_1))$ and $(c_2, f(c_2))$ belonging to $\varphi(f)$ where $c_1 \leqslant c_2$. Since f is order-preserving, $f(c_1) \leqslant f(c_2)$. Thus the strand-labeling of $\varphi(f)$ is good so $\varphi(f) \in \mathscr{L}(\mathcal{P}_d)$.

Lastly, we have that

$$\chi_1(\varphi(f)) = \chi_1(\{(c(i_\ell, j_\ell), f(c(i_\ell, j_\ell)))\}_{\ell\in[n]}) = f$$

and

$$\varphi(\chi_1(\{(c(i_\ell, j_\ell), \ell)\}_{\ell\in[n]})) = \varphi(f(c(i_\ell, j_\ell)) := \ell) = \{(c(i_\ell, j_\ell), \ell)\}_{\ell\in[n]}$$

so $\varphi = \chi_1^{-1}$. Thus χ_1 is a bijection. □

6. Applications

Here we showcase some interesting results that follow easily from our main theorems.

6.1. **Labeled trees.** In [SW86, p. 67], Stanton and White gave a nonpositive formula for the number of vertex-labeled trees with a fixed number of leaves. By connecting our work with that of Goulden and Yong [GY02], we obtain a positive expression for this number. Here we consider diagrams in $\mathcal{D}_{n,\epsilon}$ where $\epsilon = (-,\ldots,-)$ or $\epsilon = (+,\ldots,+)$. We regard these as chord diagrams to make clear the connection between our work and that of [GY02].

Theorem 6.1. Let $T_{n+1}(r) := \{\text{trees on } [n+1] \text{ with } r \text{ leaves}\}$ and $\mathcal{D}_{n,\epsilon} := \{\text{diagrams } d = \{c(i_\ell, j_\ell)\}_{\ell\in[n]}\}$. Then

$$\#T_{n+1}(r) = \sum_{d\in\mathcal{D}_{n,\epsilon}:\ d \text{ has } r \text{ chords } c(i_j, i_j+1)} \#\mathscr{L}(\mathcal{P}_d).$$

Proof. Observe that

$$\begin{aligned}\sum_{\substack{d\in\mathcal{D}_{n,\epsilon}:\ d \text{ has } r\\ \text{chords } c(i_j,i_j+1)}} \#\mathscr{L}(\mathcal{P}_d) &= \sum_{\substack{d\in\mathcal{D}_{n,\epsilon}:\ d \text{ has } r\\ \text{chords } c(i_j,i_j+1)}} \#\{\text{good labelings of } d\}\\ &= \#\left\{d(n)\in\mathcal{D}_{n,\epsilon}(n):\ \begin{array}{l} d(n) \text{ has } r \text{ chords } c(i_j,i_j+1)\\ \text{for some } i_1,\ldots,i_r\in[0,n]\end{array}\right\}\end{aligned}$$

where we consider $i_j + 1 \bmod n+1$. By [GY02, Theorem 1.1], we have a bijection between diagrams $d \in \mathcal{D}_{n,\epsilon}$ with r chords of the form $c(i_j, i_j+1)$ for some $i_1,\ldots,i_r \in [0,n]$ with good labelings and elements of $T_{n+1}(r)$. □

Corollary 6.2. We have $(n+1)^{n-1} = \sum_{d\in\mathcal{D}_{n,\epsilon}} \#\mathscr{L}(\mathcal{P}_d)$.

Proof. Let $T_{n+1} := \{\text{trees on [n+1]}\}$. One has that

$$\begin{aligned}(n+1)^{n-1} &= \#T_{n+1}\\ &= \sum_{r\geqslant 0}\#T_{n+1}(r)\\ &= \sum_{r\geqslant 0}\sum_{\substack{d\in\mathcal{D}_{n,\epsilon}:\ d \text{ has } r\\ \text{chords } c(i_j,i_j+1)}} \#\mathscr{L}(\mathcal{P}_d) \quad (\text{by Theorem 6.1})\\ &= \sum_{d\in\mathcal{D}_{n,\epsilon}}\#\mathscr{L}(\mathcal{P}_d).\end{aligned}$$

□

6.2. **Reddening sequences.** In [Kel12], Keller proves that for any quiver Q, any two reddening mutation sequences applied to $\widehat{Q}$ produce isomorphic ice quivers. As mentioned in [Kel13], his proof is highly dependent on representation theory and geometry, but the statement is purely combinatorial–we give a combinatorial proof of this result for the linearly-ordered quiver Q.

Let $R \in EG(\widehat{Q})$. A mutable vertex $i \in R_0$ is called **green** if there are no arrows $j \to i$ in R with $j \in [n+1, m]$. Otherwise, i is called **red**. A sequence of mutations $\mu_{i_r} \circ \cdots \circ \mu_{i_1}$ is **reddening** if all mutable vertices of the quiver $\mu_{i_r} \circ \cdots \circ \mu_{i_1}(\widehat{Q})$ are red. Recall that an isomorphism of quivers that fixes the frozen vertices is called a **frozen isomorphism**. We now state the theorem.

Theorem 6.3. If $\mu_{i_r} \circ \cdots \circ \mu_{i_1}$ and $\mu_{j_s} \circ \cdots \circ \mu_{j_1}$ are two reddening sequences of $\widehat{Q}_\epsilon$ for some $\epsilon \in \{+,-\}^{n+1}$, then there is a frozen isomorphism $\mu_{i_r} \circ \cdots \circ \mu_{i_1}(\widehat{Q}_\epsilon) \cong \mu_{j_s} \circ \cdots \circ \mu_{j_1}(\widehat{Q}_\epsilon)$.

Proof. Let $\mu_{i_r} \circ \cdots \circ \mu_{i_1}$ be any reddening sequence. Denote by C the **c**-matrix of $\mu_{i_r} \circ \cdots \circ \mu_{i_1}(\widehat{Q}_\epsilon)$. By Corollary 3.15, C corresponds to an oriented strand diagram $\overrightarrow{d}_C \in \overrightarrow{\mathcal{D}}_{n,\epsilon}$ with all chords of the form $\overrightarrow{c}(j,i)$ for some i and j satisfying $i < j$. As $\overrightarrow{d}_C$ avoids the configurations described in Defintion 3.13, we conclude that $\overrightarrow{d}_C = \{\overrightarrow{c}(i,i-1)\}_{i\in[n]}$ and $C = -I_n$. Since **c**-matrices are in bijection with ice quivers in $EG(\widehat{Q}_\epsilon)$ (see [NZ12, Thm 1.2]) and since $\check{Q}_\epsilon$ is an ice quiver in $EG(\widehat{Q}_\epsilon)$ whose **c**-matrix is $-I_n$, we obtain the desired result. □

6.3. **Noncrossing partitions and exceptional sequences.** In this section, we give a combinatorial proof of Ingalls' and Thomas' result that complete exceptional sequences are in bijection with maximal chains in the lattice of noncrossing partitions [IT09]. We remark that their result is more general than that which we present here. Throughout this section, we assume that Q_ϵ has $\epsilon = (-,\ldots,-)$ and we regard the strand diagrams of Q_ϵ as chord diagrams.

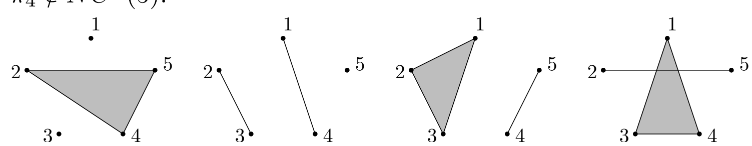
A **partition** of $[n]$ is a collection $\pi = \{B_\alpha\}_{\alpha\in I} \in 2^{[n]}$ of subsets of $[n]$ called **blocks** that are nonempty, pairwise disjoint, and whose union is $[n]$. We denote the lattice of set partitions of $[n]$ by Π_n. A set partition $\pi = \{B_\alpha\}_{\alpha\in I} \in \Pi_n$ is called **noncrossing** if for any $i < j < k < \ell$ where $i,k \in B_{\alpha_1}$ and $j,\ell \in B_{\alpha_2}$, one has $B_{\alpha_1} = B_{\alpha_2}$. We denote the lattice of noncrossing partitions of $[n]$ by $NC^{\mathbb{A}}(n)$.

Label the vertices of a convex n-gon $\mathcal{S}$ with elements of $[n]$ so that reading the vertices of $\mathcal{S}$ counterclockwise determines an increasing sequence mod n. We can thus regard $\pi = \{B_\alpha\}_{\alpha\in I} \in NC^{\mathbb{A}}(n)$ as a collection of convex hulls B_α of vertices of $\mathcal{S}$ where B_α has empty intersection with any other block $B_{\alpha'}$.

Let $n = 5$. The following partitions all belong to Π_5, but only $\pi_1, \pi_2, \pi_3 \in NC^{\mathbb{A}}(5)$.

$$\pi_1 = \{\{1\},\{2,4,5\},\{3\}\}, \pi_2 = \{\{1,4\},\{2,3\},\{5\}\}, \pi_3 = \{\{1,2,3\},\{4,5\}\}, \pi_4 = \{\{1,3,4\},\{2,5\}\}$$

Below we represent the partitions $\pi_1, \ldots, \pi_4$ as convex hulls of sets of vertices of a convex pentagon. We see from this representation that $\pi_4 \notin NC^{\mathbb{A}}(5)$.



Theorem 6.4. Let $k \in [n]$. There is a bijection between $\mathcal{D}_{k,\epsilon}(k)$ and the following chains in $NC^{\mathbb{A}}(n+1)$

$$\left\{ \{\{i\}\}_{i\in[n+1]} < \pi_1 < \cdots < \pi_k \in (NC^{\mathbb{A}}(n+1))^{k+1} : \begin{array}{c} \pi_j = (\pi_{j-1}\backslash\{B_\alpha, B_\beta\}) \sqcup \{B_\alpha \sqcup B_\beta\} \\ \text{for some } B_\alpha \neq B_\beta \text{ in } \pi_{j-1} \end{array} \right\}.$$

In particular, when $k = n$, there is a bijection between $\mathcal{D}_{n,\epsilon}(n)$ and maximal chains in $NC^{\mathbb{A}}(n+1)$. We remark that each chain described above is **saturated** (i.e. each inequality appearing in $\{\{i\}\}_{i\in[n+1]} < \pi_1 < \cdots < \pi_k$ is a covering relation).

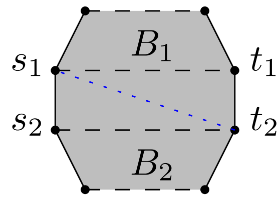
Proof. Let $d(k) = \{(c(i_\ell, j_\ell), \ell)\}_{\ell\in[k]} \in \mathcal{D}_{k,\epsilon}(k)$. Define $\pi_{d(k),1} := \{\{i\}\}_{i\in[n+1]} \in \Pi_{n+1}$. Next, define $\pi_{d(k),2} := \left(\pi_{d(k),1}\backslash\{\{i_1+1\},\{j_1+1\}\}\right) \sqcup \{i_1+1, j_1+1\}$. Now assume that $\pi_{d(k),s}$ has been defined for some $s \in [k]$. Define $\pi_{d(k),s+1}$ to be the partition obtained by merging the blocks of $\pi_{d(k),s}$ containing i_s+1 and j_s+1. Now define $f(d(k)) := \{\pi_{d(k),s} : s \in [k+1]\}$.

It is clear that $f(d(k))$ is a chain in Π_{n+1} with the desired property as $\pi_1 \lessdot \pi_2$ in Π_{n+1} if and only if π_2 is obtained from π_1 by merging exactly two distinct blocks of π_1. To see that each $\pi_{d(k),s} \in NC^{\mathbb{A}}(n+1)$ for all $s \in [k+1]$, suppose a crossing of two blocks occurs in a partition appearing in $f(d(k))$. Let $\pi_{d(k),s}$ be the smallest partition of $f(d(k))$ (with respect to the partial order on set partitions) with two blocks crossing blocks B_1 and B_2. Without loss of generality, we assume that $B_2 \in \pi_{d(k),s}$ is obtained by merging the blocks $B_{\alpha_1}, B_{\alpha_2} \in \pi_{d(k),s-1}$ containing $i_{s-1}+1$ and $j_{s-1}+1$, respectively. This means that $d(k)$ has a chord $c(i_{s-1}, j_{s-1})$ that crosses at least one other chord of $d(k)$. This contradicts that $d(k) \in \mathcal{D}_{k,\epsilon}(k)$. Thus $f(d(k))$ is a chain in $NC^{\mathbb{A}}(n+1)$ with the desired property.

Next, we define a map g that is the inverse of f. Let $C = (\pi_1 = \{\{i\}\}_{i\in[n+1]} < \pi_2 < \cdots < \pi_{k+1}) \in (NC^{\mathbb{A}}(n+1))^{k+1}$ be a chain where each partition in C satisfies $\pi_j = (\pi_{j-1}\backslash\{B_\alpha, B_\beta\}) \sqcup \{B_\alpha \sqcup B_\beta\}$ for some $B_\alpha \neq B_\beta$ in π_{j-1}. As $\pi_2 = (\pi_1\backslash\{\{s_1\},\{t_1\}\}) \sqcup \{s_1, t_1\}$, define $c(i_1, j_1) := c(s_1-1, t_1-1)$ where we consider $s_1 - 1$ and $t_1 - 1$ mod $n+1$. Assume $s_1 < t_1$. If t_1 is in a block of size 3 in π_3, let t denote the element of this block where $t \neq s_1, t_1$. If t satisfies $s_1 < t < t_1$, define $c(i_2, j_2) := c(s_1-1, t-1)$. Otherwise, define $c(i_2, j_2) := c(t_1-1, t-1)$. If there is no block of size 3 in π_3, define $c(i_2, j_2) := c(s_2-1, t_2-1)$ where $\{s_2\}$ and $\{t_2\}$ were singleton blocks in π_2 and $\{s_2, t_2\}$ is a block in π_3.

Now suppose we have defined $c(i_r, j_r)$. Let B denote the block of π_{r+2} obtained by merging two blocks of π_{r+1}. If B is obtained by merging two singleton blocks $\{s_{r+1}\}, \{t_{r+1}\} \in \pi_{r+1}$, define $c(i_{r+1}, j_{r+1}) := c(s_{r+1}-1, t_{r+1}-1)$.

Otherwise, $B = B_1 \sqcup B_2$ where $B_1, B_2 \in \pi_{r+1}$. Now note that, up to rotation and up to adding or deleting elements of $[n+1]$ for B_1 and B_2, $B_1 \sqcup B_2$ appears in π_{r+2} as follows.



Thus define $c(i_{r+1}, j_{r+1}) := c(s_1 - 1, t_2 - 1)$. Finally, put $g(C) := \{(c(i_\ell, j_\ell), \ell) : \ell \in [k]\}$.

We claim that $g(C)$ has no crossing chords. Suppose $(c(s_i, t_i), i)$ and $(c(s_j, t_j), j)$ are crossing chords in $g(C)$ with $i < j$ and $i, j \in [k]$. We further assume that

$$j = \min\{j' \in [i+1, k] : (c(s_{j'}, t_{j'}), j') \text{ crosses } (c(s_i, t_i), i) \text{ in } g(C)\}.$$

We observe that $s_i + 1, t_i + 1 \in B_1$ for some block $B_1 \in \pi_j$ and that $s_j + 1, t_j + 1 \in B_2$ for some block $B_2 \in \pi_{j+1}$. We further observe that $s_j + 1, t_j + 1 \notin B_1$ otherwise, by the definition of the map g, the chords $(c(s_i, t_i), i)$ and $(c(s_j, t_j), j)$ would be noncrossing. Thus $B_1, B_2 \in \pi_{j+1}$ are distinct blocks that cross so $\pi_{j+1} \notin NC^{\mathbb{A}}(n+1)$. We conclude that $g(C)$ has no crossing chords so $g(C) \in \mathcal{D}_{k,\epsilon}(k)$.

To complete the proof, we show that $g \circ f = 1_{\mathcal{D}_{k,\epsilon}(k)}$. The proof that $f \circ g$ is the identity map is similar. Let $d(k) \in \mathcal{D}_{k,\epsilon}(k)$. Then $f(d(k)) = \{\{i\}\}_{i \in [n+1]} < \pi_1 < \cdots < \pi_k$ where for any $s \in [k]$ we have

$$\pi_s = (\pi_{s-1} \backslash \{B_\alpha, B_\beta\}) \sqcup \{B_\alpha, B_\beta\}$$

where $i_{s-1} + 1 \in B_\alpha$ and $j_{s-1} + 1 \in B_\beta$. Then we have

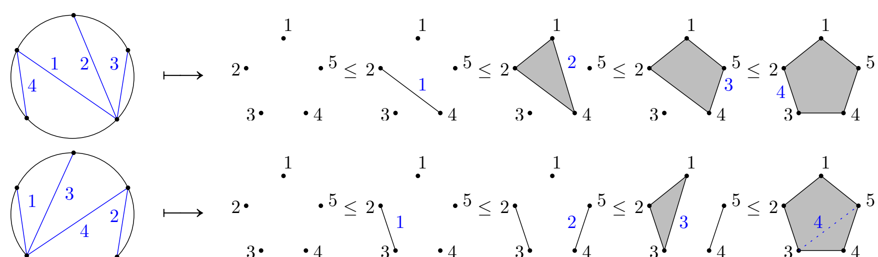
$$g(f(d(k))) = \{c((i_\ell + 1) - 1, (j_\ell + 1) - 1), \ell)\}_{\ell \in [k]} = \{(c(i_\ell, j_\ell), \ell)\}_{\ell \in [k]}.$$

□

Corollary 6.5. If $\epsilon = (-, \ldots, -)$, then the exceptional sequences of Q_ϵ are in bijection with saturated chains in $NC^{\mathbb{A}}(n+1)$ of the form

$$\left\{ \{\{i\}\}_{i \in [n+1]} < \pi_1 < \cdots < \pi_k \in (NC^{\mathbb{A}}(n+1))^{k+1} : \begin{array}{c} \pi_j = (\pi_{j-1} \backslash \{B_\alpha, B_\beta\}) \sqcup \{B_\alpha \sqcup B_\beta\} \\ \text{for some } B_\alpha \neq B_\beta \text{ in } \pi_{j-1} \end{array} \right\}.$$

Example 6.6. Here we give examples of the bijection from the previous theorem with $k = 4$.



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