Two problems about hyperplane arrangements (secretly about matroids)

Combinatorics meets algebra and topology

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Two problems about hyperplane arrangements:

- The "Top-Heavy Conjecture"
- The non-negativity of the coefficients of Kazhdan–Lusztig polynomials

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I will tell the story using hyperplane arrangements (realizable matroids), but both questions make sense for arbitrary matroids.

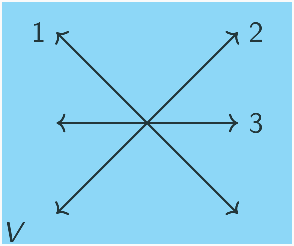
Problem 1 - The Top-Heavy Conjecture

Hyperplane arrangements and flats

V - vector space, $\mathcal{A}$ - finite set of hyperplanes with $\bigcap_{H \in \mathcal{A}} H = \{0\}$.

A **flat** is a subspace obtained by intersecting some of the hyperplanes.

Three lines in $\mathbb{R}^2$



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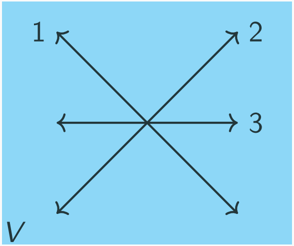

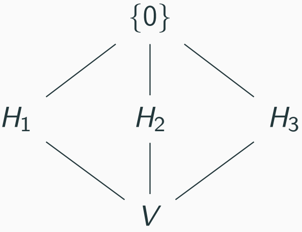

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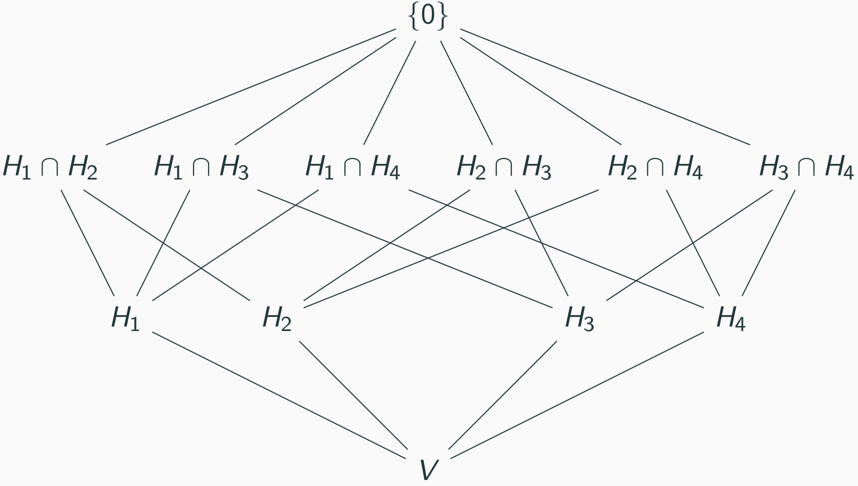

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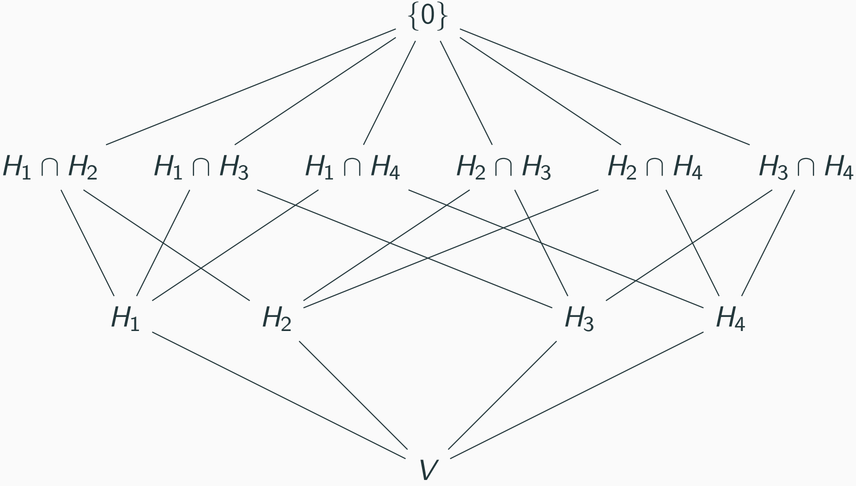

$6 \geq 4$

The Top-Heavy Conjecture

Problem 1: The Top-Heavy Conjecture

Conjecture (Dowling–Wilson 1974)

For all $k \leq \frac{1}{2} \dim V$*, we have*

$$\#(\textit{flats of } \dim k) \geq \#(\textit{flats of } \operatorname{codim} k).$$

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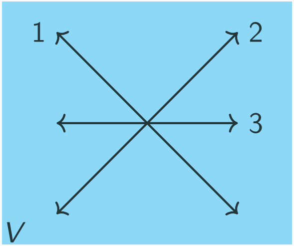
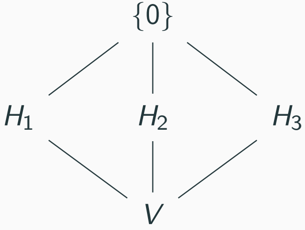
Problem 2 - The non-negativity of the coefficients of Kazhdan–Lusztig polynomials

Characteristic polynomial of a hyperplane arrangement





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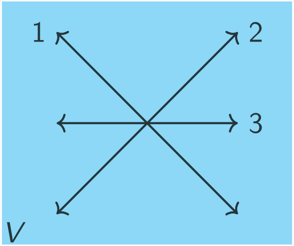

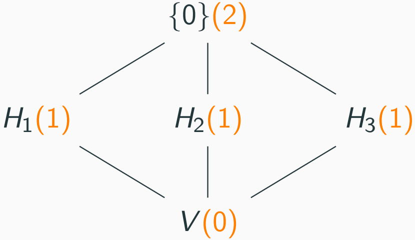

- $\mathrm{codim} : L(\mathcal{A}) \to \mathbb{Z}_{\geq 0}$

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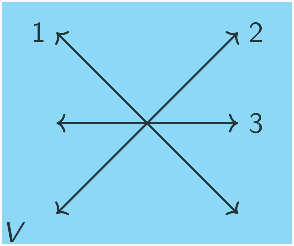

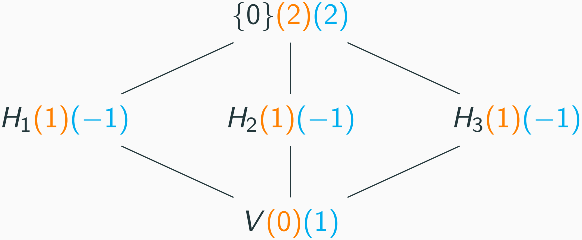

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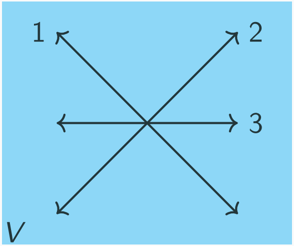

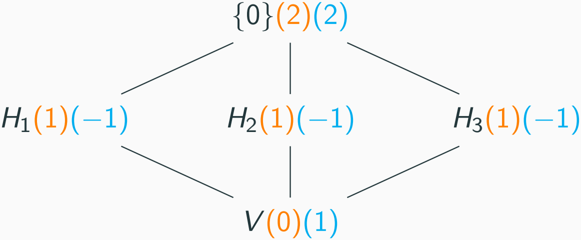

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Definition

The **characteristic polynomial** of $\mathcal{A}$ is

$$\chi_{\mathcal{A}}(t) = \sum_{F \in L(\mathcal{A})} \mu(F) t^{\dim F}.$$

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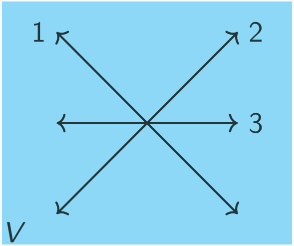

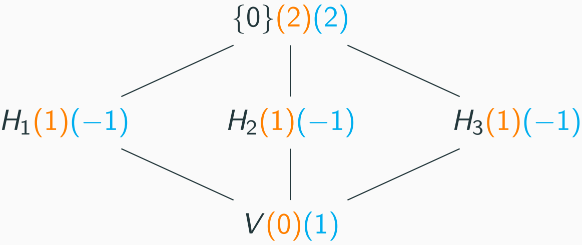

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$$\chi_{\mathcal{A}}(t) = t^2 - 3t + 2$$

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Localization and contraction of arrangements

For any flat $F \in L(\mathcal{A})$, we have two new arrangements:

- $\mathcal{A}^F$ - the **localization of $\mathcal{A}$ at F** (arrangement in V/F).
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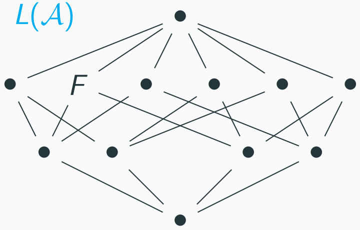

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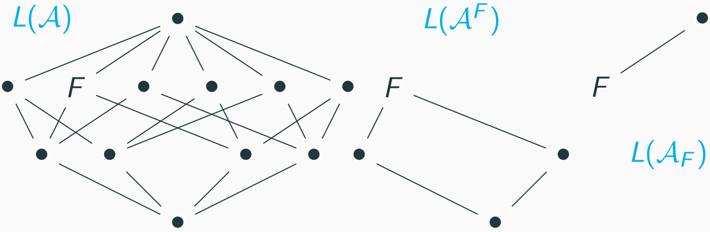

Definition of Kazhdan–Lusztig polynomials

Definition (Elias–Proudfoot–Wakefield 2016)

To each arrangement $\mathcal{A}$, we have a unique polynomial $P_{\mathcal{A}}(t) \in \mathbb{Z}[t]$ such that

- If $\dim V = 0$, then $P_{\mathcal{A}}(t) = 1$.

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What do these polynomials look like?

Three lines in $\mathbb{R}^2$: $P_{\mathcal{A}}(t) = 1$.

Four planes in $\mathbb{R}^3$: $P_{\mathcal{A}}(t) = 1 + 2t$.

Examples [Elias–Proudfoot–Wakefield–Young 2016]

KL polynomials for the arrangement of $d+1$ generic hyperplanes in d-space.

$d =$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
t			2	5	9	14	20	27	35	44	54	65
t^2					5	21	56	120	225	385	616	936
t^3							14	84	300	825	1925	4004
t^4									42	330	1485	5005
t^5											132	1287

Examples [Elias–Proudfoot–Wakefield–Young 2016]

KL polynomials for the type A_n Coxeter arrangement.

$n =$	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
t				1	5	16	42	99	219	466
t^2						15	175	1225	6769	32830
t^3								735	16065	204400
t^4										76545

Properties

Problem 2: KL polynomials have non-negative coefficients

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A sequence $a_0, \ldots, a_r$ is called **log-concave** if for all $1 < i < r$, we have $a_{i-1}a_{i+1} \leq a_i^2$. The sequence has **no internal zeroes** if $\{i \mid a_i \neq 0\}$ is an interval.

The Proofs - Problems 1 and 2

Combinatorics meets topology

Reminder

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Theorem (Elias–Proudfoot–Wakefield 2016)

For any hyperplane arrangement $\mathcal{A}$, *the KL polynomial* $P_{\mathcal{A}}(t)$ *has non-negative coefficients.*

Some geometry

We have

$$V \hookrightarrow \bigoplus_{H \in \mathcal{A}} V/H \cong \bigoplus_{H \in \mathcal{A}} \mathbb{A}^1 \subset \prod_{H \in \mathcal{A}} \mathbb{P}^1.$$

Let $Y := \overline{V} \subset \prod_{H \in \mathcal{A}} \mathbb{P}^1$. $\longleftarrow$ the Schubert variety of $\mathcal{A}$

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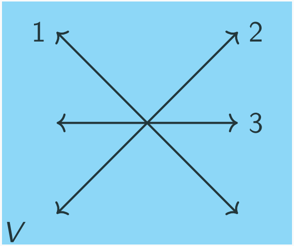
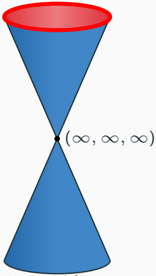


neighborhood of $(0, 0, 0)$



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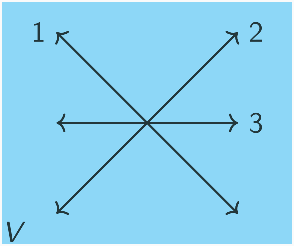

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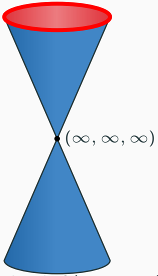

neighborhood of (∞, ∞, ∞)

Y has a stratification $Y = \coprod_F Y_F$ by affine spaces.

Affine pavings

The stratification by affine cells gives us two things:

1. $\dim H^{2k}(Y) = \#(\text{flats of } \operatorname{codim} k)$.
2. [Björner–Ekedahl 2009] There is an injection

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One property of intersection cohomology:

- $\mathrm{IH}^\bullet(Y)$ satisfies Hard Lefschetz (since Y is projective).

Proof of the Top-Heavy Conjecture

Let $L \in H^2(Y)$ be an ample class. If $k \leq \frac{1}{2} \dim V$, then consider the following diagram.

$$\begin{array}{ccc} H^{2(\dim V - k)}(Y) & \xhookrightarrow{\text{B–E 09}} & \mathrm{IH}^{2(\dim V - k)}(Y) \\ {\scriptstyle L^{2(\dim V - 2k)}}\uparrow & & \cong \uparrow {\scriptstyle L^{2(\dim V - 2k)}} \\ H^{2k}(Y) & \xhookrightarrow[\text{B–E 09}]{} & \mathrm{IH}^{2k}(Y) \end{array}$$

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$\Longrightarrow$ Top-Heavy Conjecture (We gave a proof for Problem 1!)

“Proof” for Problem 2

Theorem (Elias–Proudfoot–Wakefield 2016)

For any hyperplane arrangement $\mathcal{A}$, we have

$$P_{\mathcal{A}}(t) = \sum_{i \geq 0} t^i \dim \mathrm{IH}^{2i}_{(\infty,\ldots,\infty)}(Y).$$

Problems 1 and 2 for arbitrary matroids

Combinatorics meets algebra (while being informed by topology)

Matroids

A **matroid** ([Whitney 1935]) is a gadget that generalizes the notion of linear (in)dependence in a vector space. It has a

- ground set I (finite set)
- a collection of distinguished subsets (independent sets, bases, closed sets, circuits, $\ldots$) satisfying some axioms

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Examples can be gotten from

- vectors in a vector space
- hyperplane arrangements
- graphs

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Not all matroids can be realized as vectors in a vector space.

Problems 1 and 2 for arbitrary matroids

Problem 1: The Top-Heavy Conjecture

Conjecture (Dowling–Wilson 1974)

Let M be an arbitrary matroid. For all $k \leq \frac{1}{2}\mathrm{rk}M$, we have

$$\#L(M)^{\mathrm{rk}M-k} \geq \#L(M)^{k}.$$

Problem 2: KL polynomials have non-negative coefficients

Conjecture (Elias–Proudfoot–Wakefield 2016)

For any matroid M, the KL polynomial $P_M(t)$ has non-negative coefficients.

Repeated slide! Proof of the Top-Heavy Conjecture

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$\Longrightarrow$ Top-Heavy Conjecture for realizable matroids (hyperplane arrangements)

The semi-wonderful model (in progress: Braden–Huh–M.–Proudfoot–Wang)

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- [Huh–Wang 2017] There is a ring $B^\bullet(M)$ such that $B^\bullet(M) \cong H^\bullet(Y)$ when M is realizable.
- [Braden–Huh–M.–Proudfoot–Wang] There is a ring $A^\bullet(M)$ such that $A^\bullet(M) \cong H^\bullet(\widetilde{Y})$ when M is realizable.

Strategy for the proof (in progress: Braden–Huh–M.–Proudfoot–Wang)

Note that

$$H^\bullet(Y) \subset \mathrm{IH}^\bullet(Y) \subset H^\bullet(\widetilde{Y}).$$

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1. Decompose $A^\bullet(M)$ as a $B^\bullet(M)$-module.
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4. Run the same argument.

$$\begin{array}{ccc} B^{\mathrm{rk}M-k}(M) & \hookrightarrow & I^{\mathrm{rk}M-k}(M) \\ \uparrow & & \cong\uparrow HL \\ B^k(M) & \hookrightarrow & I^k(M) \end{array}$$

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Strategy:

1. Decompose $A^\bullet(M)$ as a $B^\bullet(M)$-module.
2. Find the summand $I^\bullet(M)$, and get injection $B^\bullet(M) \hookrightarrow I^\bullet(M)$.
3. Prove "Hard Lefschetz" for $I^\bullet(M)$.
4. Run the same argument.

$$\begin{array}{ccc} B^{\mathrm{rk}M-k}(M) & \hookrightarrow & I^{\mathrm{rk}M-k}(M) \\ \Big\uparrow & & \cong \Big\uparrow HL \\ B^k(M) & \hookrightarrow & I^k(M) \end{array}$$

Problem 2 for arbitrary matroids
(in progress: Braden–Huh–M.–Proudfoot–Wang)

Conjecture (Braden–Huh–M.–Proudfoot–Wang)

For an arbitrary matroid M, we have

$$P_M(t) = \mathrm{Poin}(I^\bullet(M) \otimes_{B^\bullet(M)} \mathbb{C}).$$

Thanks!