Singular Hodge theory of matroids

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Goals

- Prove the top heavy conjecture for arbitrary matroids.
- Prove that Kazhdan–Lusztig polynomials of matroids have non-negative coefficients.

Kazhdan–Lusztig theory

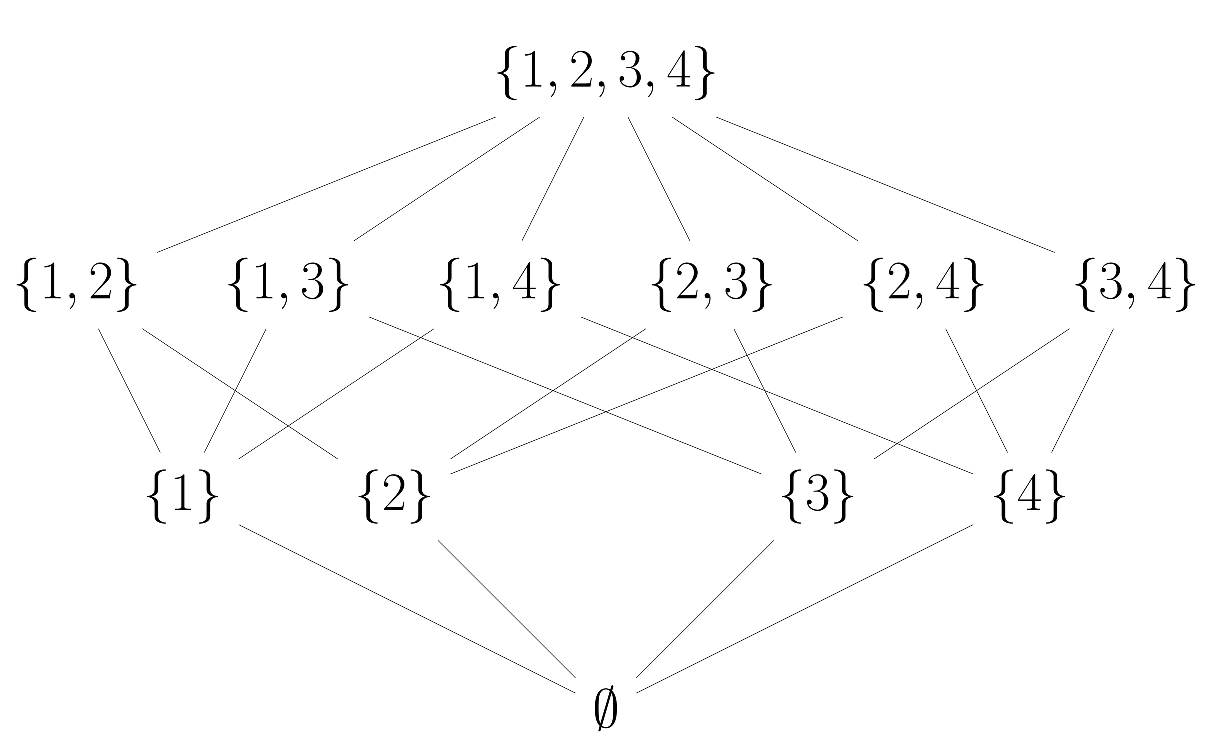
For Coxeter groups	For matroids
Coxeter group	matroid M (with ground set E)
Weyl group	realizable matroid
Bruhat poset	lattice of flats $L(M)$
R-polynomial	characteristic polyn. $\chi_M(t) = \sum_{F \in L(M)} \mu(F) t^{\mathrm{rk}M - \mathrm{rk}F}$
Hecke algebra	?
Polo	real-rooted
Schubert variety $\overline{X_w}$	$Y := \overline{V} \subset (\mathbb{P}^1)^E$

Definition [Elias–Proudfoot–Wakefield 2016]

To each matroid M, we have a unique polynomial $P_M(t) \in \mathbb{Z}[t]$ such that

- If $\mathrm{rk}M = 0$, then $P_M(t) = 1$.
- If $\mathrm{rk}M > 0$, then $\deg P_M(t) < \frac{1}{2}\mathrm{rk}M$.
- For every M, $t^{\mathrm{rk}M} P_M(t^{-1}) = \sum_{F \in L(M)} \chi_{M_F}(t) P_{M^F}(t)$.

Example: $U_{3,4}$



$$\chi_M(t) = t^3 - 4t^2 + 6t - 3$$
$$P_M(t) = 1 + 2t$$
$$4 \leq 6$$

The realizable case

Theorem [Elias–Proudfoot–Wakefield 2016]

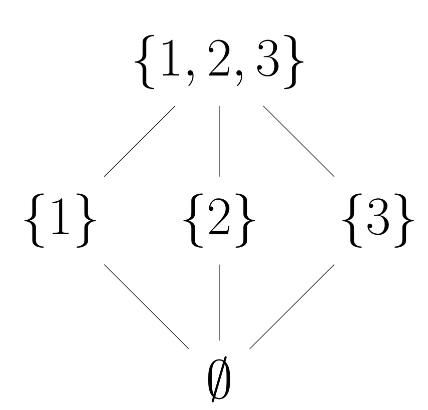
For every realizable M,
$$P_M(t) = \sum_{i \geq 0} \dim \mathrm{IH}^{2i}_{(\infty,\ldots,\infty)}(Y) t^i.$$

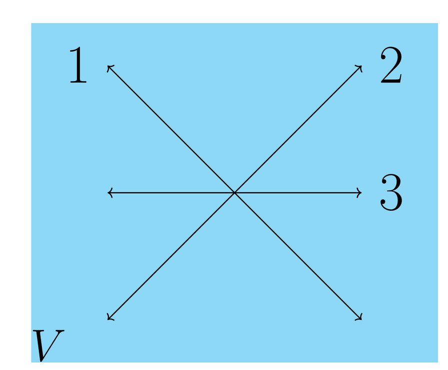
Conjecture [Dowling–Wilson 1974], Theorem [Huh–Wang 2017] for M realizable

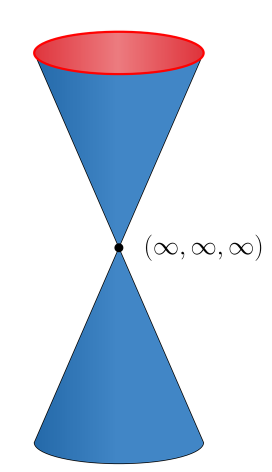
For all $k \leq \frac{1}{2}\mathrm{rk}M$, we have
$$\#L(M)^k \leq \#L(M)^{\mathrm{rk}M - k}.$$

$$V \hookrightarrow \bigoplus_{H \in \mathcal{A}} V/H \cong \bigoplus_{H \in \mathcal{A}} \mathbb{A}^1 \subset \prod_{H \in \mathcal{A}} \mathbb{P}^1$$

Let $Y := \overline{V} \subset \prod_{H \in \mathcal{A}} \mathbb{P}^1$.







Y has a stratification $Y = \coprod_{F \in L(M)} Y_F$ by affine spaces.

$Y_F = \{p \in Y \mid p_i = \infty \iff i \notin F\}$.

Example: $Y_\emptyset = \{(\infty, \ldots, \infty)\}$ and $Y_E = V$.

Properties of this affine paving

1. $\dim H^{2k}(Y) = \#L(M)^k$.
2. [Björner–Ekedahl 2009] There is an injection $$H^\bullet(Y) \hookrightarrow \mathrm{IH}^\bullet(Y).$$

Proof of the top-heavy conjecture

Let $L \in H^2(Y)$ be an ample class. If $k \leq \frac{1}{2}\mathrm{rk}M$, then consider the following diagram.

$$\begin{array}{ccc} H^{2(\mathrm{rk}M-k)}(Y) & \xrightarrow{\text{B–E 09}} & \mathrm{IH}^{2(\mathrm{rk}M-k)}(Y) \\ {\scriptstyle L^{2(\mathrm{rk}M-2k)}} \uparrow & & \uparrow {\scriptstyle \cong\, L^{2(\mathrm{rk}M-2k)}} \\ H^{2k}(Y) & \xrightarrow[\text{B–E 09}]{} & \mathrm{IH}^{2k}(Y) \end{array}$$

Arbitrary matroids [Braden–Huh–M.–Proudfoot–Wang]

Define the "semi-wonderful" resolution
$$\widetilde{Y} \longrightarrow Y \qquad \text{by}$$

1. first blowing up the point $Y_\emptyset$,
2. then the proper transforms of $Y_{\{i\}}$,
3. then the proper transforms of Y_F (with $\mathrm{rk}F = 2$) strata, and so on...

Theorems [Huh–Wang 2017], [BHMPW]

- There is a ring $B^\bullet(M) \cong H^\bullet(Y)$ when M is realizable.
- There is a ring $A^\bullet(M) \cong H^\bullet(\widetilde{Y})$ when M is realizable.

Sketch of top-heavy conjecture for all matroids

Note that $H^\bullet(Y) \subset \mathrm{IH}^\bullet(Y) \subset H^\bullet(\widetilde{Y})$.

Strategy:

1. Decompose $A^\bullet(M)$ as a $B^\bullet(M)$-module.
2. Find the summand $I^\bullet(M)$, and make an injection $B^\bullet(M) \overset{PD}{\hookrightarrow} I^\bullet(M)$.
3. Prove Hard Lefschetz for $I^\bullet(M)$ and run the same argument.

$$\begin{array}{ccc} B^{\mathrm{rk}M-k}(M) & \overset{PD}{\hookrightarrow} & I^{\mathrm{rk}M-k}(M) \\ \uparrow & & \uparrow {\scriptstyle \cong\, HL} \\ B^k(M) & \underset{PD}{\hookrightarrow} & I^k(M) \end{array}$$

Definition of semi-wonderful Chow ring $A^\bullet(M)$

$A^\bullet(M)$ is the quotient of
$$\mathbb{C}[x_F, y_i \mid F \in L(M) \text{ is a proper flat, and } i \in E]$$
by the ideal generated by

- $x_{F_1} x_{F_2}$, where F_1 and F_2 are incomparable,
- $y_i - \sum_{i \notin F} x_F$, and
- $y_i x_F$ if $i \notin F$.

$B^\bullet(M)$ is the subring of $A^\bullet(M)$ generated by the y_i, for all $i \in E$.

Conjecture (non-negativity of the $P_M(t)$)

$$P_M(t) = \mathrm{Poin}(I^\bullet(M) \otimes_{B^\bullet(M)} \mathbb{C}).$$