A CONTINUOUS ASSOCIAHEDRON OF TYPE A

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ABSTRACT. Taking a representation-theoretic viewpoint, we construct a continuous associahedron motivated by the realization of the generalized associahedron in the physical setting. We show that our associahedron shares important properties with the generalized associahedron of type A. Our continuous associahedron is convex and manifests a cluster theory: the points which correspond to the clusters are on its boundary, and the edges that correspond to mutations are given by intersections of hyperplanes. This requires development of several methods that are continuous analogues of discrete methods. We conclude the paper by showing that there is a sequence of embeddings of type A generalized associahedra into our continuous associahedron.

CONTENTS

1. INTRODUCTION

The classical $(n-2)$-dimensional associahedron is a convex polytope whose vertices are binary bracketings of words using n symbols, and whose edges correspond to one application of associativity. It was first discovered by Tamari while studying general questions about associativity in algebras [T51] and rediscovered under the name of Stasheff polytope in the context of homotopy theory [S63]. The associahedron captures the combinatorial structure of a variety of objects throughout mathematics, including triangulations of polygons, operads and homotopy theory, real moduli spaces, and cluster structures [FZ01, CFZ02]. For details of these constructions and for other realizations of associahedra, we point to [CSZ15] and the references therein.

The purpose of this paper is to introduce a continuous version of the classical associahedron guided by recent advances in cluster categories and particle physics. Our motivation for such a construction is two-fold:

- From [FZ01, CFZ02], the associahedron can be viewed as the "cluster polytope" which captures the combinatorics of type A cluster algebras. In each of [IT15] and [IRT20], the authors use representation theory of type A quivers with infinitely many vertices to give an analogue for a cluster structure. However, neither of these settings has a known cluster algebra. In this paper, we strengthen the analogy between the continuous and finite dimensional cases by developing an analogue of the cluster polytope in the continuous setting.

- In [AB+18], where the scattering amplitude of a certain quantum field theory is treated, the authors realized the corresponding amplituhedron as the generalized associahedron of type A. Moreover, a continuous associahedron of type A appears in [AH+19], where it is obtained as an inverse limit of the generalized associahedra for A_n, as n goes to infinity. In contrast, we start from a category with a continuous cluster structure, allowing us to treat time as a continuous phenomenon from the beginning and consequently give a continuous associahedron without taking limits.

As in [AH+19], our construction of the continuous associahedron is motivated by the representation-theoretic techniques developed in [BD+18].

1.1. **The continuous associahedron.** Fix an algebraically closed field k of characteristic 0. The central category considered in [IT15] is the Krull–Schmidt triangulated category $\mathcal{D}$ (defined in Section 2.2), whose indecomposable objects are the points (x, y) in $\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$, and whose shift functor is given by $(x, y)[1] := (x + \pi, -y)$ for every indecomposable object (x, y). Choosing a zigzag $\mathcal{Z}$ in $\mathcal{D}$ (Definition 3.1) determines a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ (Section 4.1). One may "quilt" off of $\mathcal{Z}$ (Definition 3.3), just as one knits off of the projective slice in $\mathcal{D}^b(A_n)$. The indecomposable objects of $\mathcal{D}^{\leq 0}$ are those obtained by quilting off $\mathcal{Z}$, and the indecomposable objects of $\mathcal{D}^{\geq 0}$ are those obtained by inverse quilting off $\mathcal{Z}[1]$. We write $\mathcal{D}^{\heartsuit} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ for the heart of this t-structure and note that it plays an analogous role to $\mathrm{rep}(A_n)$ in our continuous story. See Section 4.4 for more on this philosophy.

The setting for the continuous associahedron is the category $\mathcal{C}_{\mathcal{Z}} = \mathrm{add}\left(\mathrm{Ind}(\mathcal{D}^{\heartsuit}) \sqcup \mathrm{Ind}(\mathcal{Z}[1])\right)$. The category $\mathcal{C}_{\mathcal{Z}}$ has a continuous analogue of the mesh relations appearing in the Auslander–Reiten theory of algebras of type A: this allows us to define continuous deformed mesh relations (Definition 3.18) using a function $\underline{c} : \mathrm{Ind}(\mathcal{C}_{\mathcal{Z}}) \to \mathbb{R}_{>0}$ analogous to the construction in [BD+18]. For the remainder of the introduction, we fix a zigzag $\mathcal{Z}$ and such a function $\underline{c}$.

Definition (Definition 6.5). The continuous associahedron $\mathbb{U}_{\mathcal{Z},\underline{c}}$ is the subset of $\prod_{\mathrm{Ind}(\mathcal{C}_{\mathcal{Z}})} \mathbb{R}$ consisting of nonnegative solutions of the continuous deformed mesh relations with respect to $\underline{c}$.

Theorem A (Theorem 6.7). The continuous associahedron $\mathbb{U}_{\mathcal{Z},\underline{c}}$ is convex in the sense that any line segment in $\prod_{\mathrm{Ind}(\mathcal{C}_{\mathcal{Z}})} \mathbb{R}$ whose endpoints are in $\mathbb{U}_{\mathcal{Z},\underline{c}}$ is entirely contained in $\mathbb{U}_{\mathcal{Z},\underline{c}}$.

1.2. **The continuous associahedron as a cluster "polytope".** Using the triangulated structure in $\mathcal{D}$, we define compatibility of a pair of indecomposables in $\mathcal{C}_{\mathcal{Z}}$ (Definition 5.2). A **T**-cluster $\mathcal{T}$ (Definition 5.4) is a maximal collection of pairwise compatible indecomposables in $\mathcal{C}_{\mathcal{Z}}$. We define an exchange relation $\mathcal{T} \to (\mathcal{T} \setminus \{X\}) \cup \{Y\}$, called **T**-mutation (Definition 5.10), that replaces exactly one indecomposable $X \in \mathcal{T}$ with a new indecomposable $Y \notin \mathcal{T}$ whenever possible. We say a solution Φ corresponds to a **T**-cluster $\mathcal{T}$ if $\Phi(X) = 0$ for all $X \in \mathcal{T}$. Given a **T**-cluster $\mathcal{T}$, it is not known whether a solution corresponding to $\mathcal{T}$ exists, and if it exists, whether or not it is unique. We refer the reader to Question 6.3 and the discussion preceding it for further details.

Remark. A key difference to the cluster structure in [BD+18] is that one is not necessarily able to exchange an arbitrary object in a **T**-cluster (see Example 5.12). Additionally, our category $\mathcal{C}_{\mathcal{Z}}$ is not a cluster category in the sense of [BM+06] because $\mathcal{C}_{\mathcal{Z}}$ is not an orbit category of $\mathcal{D}$. We do, however, have a cluster theory as in [IRT20, Definition 5.1.1].

Cluster polytopes were introduced by Fomin and Zelevinsky [FZ01] and their polytopality was proved by Chapoton, Fomin, and Zelevinsky [CFZ02]. In particular, the vertices correspond to clusters and the edges to mutation. We show similar behavior in $\mathbb{U}_{\mathcal{Z},\underline{c}}$. To each indecomposable X in $\mathcal{C}_{\mathcal{Z}}$, we associate the hyperplane $\mathbb{H}_X$ in $\prod_{\mathrm{Ind}(\mathcal{C}_{\mathcal{Z}})} \mathbb{R}$ given by setting the X-coordinate to 0. Additionally, using Theorem A, we define a point X to be on the boundary of $\mathbb{U}_{\mathcal{Z},\underline{c}}$ if there exists a nontrivial line segment (a line segment with distinct endpoints) ending at X that cannot be extended inside $\mathbb{U}_{\mathcal{Z},\underline{c}}$ such that X is no longer an endpoint (Definition 6.8).

Theorem B (Theorems 6.9 and 6.11)**.** If $\mathcal{T}$ is a **T**-cluster, then a solution corresponding to $\mathcal{T}$ is on the boundary of $\mathbb{U}_{\mathcal{Z},\underline{c}}$. Let $\mathcal{T}$ and $\mathcal{T}'$ be two **T**-clusters, each of which uniquely correspond to two respective points Φ and Φ' on the boundary of $\mathbb{U}_{\mathcal{Z},\underline{c}}$. Then there is a **T**-mutation $\mathcal{T} \to \mathcal{T}'$ if and only if there is an edge in $\mathbb{U}_{\mathcal{Z},\underline{c}}$ connecting Φ to Φ', which is given by $\bigcap_{X \in \mathcal{T} \cap \mathcal{T}'} \mathbb{H}_X$.

In Section 6.4, we embed the finite dimensional generalized associahedra $\mathbb{U}_{n,\underline{c}}$ given in [BD+18], for all n, into the continuous associahedron $\mathbb{U}_{\mathcal{Z},\underline{c}}$ in a way that preserves the cluster structures in the sense of the following theorem.

Theorem C (Theorem 6.18)**.** There is an infinite sequence of embeddings

$$\mathbb{U}_{2,\underline{c}} \hookrightarrow \mathbb{U}_{3,\underline{c}} \hookrightarrow \cdots \hookrightarrow \mathbb{U}_{n,\underline{c}} \hookrightarrow \mathbb{U}_{n+1,\underline{c}} \hookrightarrow \cdots \mathbb{U}_{\mathcal{Z},\underline{c}}.$$

For $\mathbb{U}_{n,\underline{c}} \hookrightarrow \mathbb{U}_{n+1,\underline{c}}$, cluster vertices are taken to cluster vertices and mutation edges are taken to mutation edges. For $\mathbb{U}_{n,\underline{c}} \hookrightarrow \mathbb{U}_{\mathcal{Z},\underline{c}}$, cluster vertices are taken to solutions corresponding to **T**-clusters and mutation edges are taken to edges corresponding to **T**-mutations.

1.3. **Outline.** In Section 2.1, we review the representation-theoretic construction of the finite dimensional associahedron in [BD+18]. In the rest of Section 2, we introduce the category $\mathcal{D}$ and prove some basic properties. Section 3 is dedicated to introducing the continuous deformed mesh relations using a process called "quilting", which is analogous to knitting in the finite dimensional case. We develop connections to representations of quivers in Section 4, where we prove that the hearts of certain t-structures have several of the same properties as finitely generated representations of type A_n quivers. In Section 5, we introduce **T**-clusters and **T**-mutation. Our last section, Section 6, is devoted to the definition of the associahedron $\mathbb{U}_{\mathcal{Z},\underline{c}}$ and the proofs of our main results.

1.4. **Setting and notation.** We work over an algebraically closed field k of characteristic 0 (for example, the field of complex numbers). For a finite dimensional k-algebra Λ, by $\mathrm{mod}\,\Lambda$ we denote the category of all finite dimensional left Λ-modules and $\mathrm{Ind}(\Lambda)$ denotes the set of all isomorphism classes of indecomposable objects in $\mathrm{mod}\,\Lambda$. To $\mathrm{mod}\,\Lambda$, we associate its Auslander–Reiten quiver Γ_Λ, where the set of vertices of Γ_Λ is in bijection with $\mathrm{Ind}(\Lambda)$ and the arrows between two vertices X and Y in Γ_Λ correspond to the irreducible morphisms in $\mathrm{Hom}_\Lambda(X, Y)$. We denote by $\mathcal{D}^b(\Lambda)$ the bounded derived category of Λ, and consider the full additive subcategory $\mathcal{C}$ of $\mathcal{D}^b(\Lambda)$ generated by $\mathrm{Ind}(\Lambda) \sqcup \{P_i[1] \mid i \in Q_0\}$, where $P_i[1]$ is the shift of the projective module P_i in $\mathcal{D}^b(\Lambda)$. For a detailed treatment of the representation theory of finite dimensional algebras, and for all undefined terms, we point to [ASS06]. In this paper, we write $\mathcal{D}^b(A_n) := \mathcal{D}^b(\Lambda)$, where $\Lambda = kA_n$ is the path algebra of the type A_n quiver.

2. Finite case and $\mathcal{D}$

2.1. **Amplituhedron for Dynkin quivers.** The amplituhedron studied by physicists in [AB+18] was constructed representation theoretically in [BD+18]. The main goal of this paper is to introduce an analogue of this construction in the continuous setting. To set the scene, we use this section to briefly recall the construction in [BD+18], preferring to stick to an example rather than inundating the reader with technical details.

Example 2.1. Let Q be the quiver given by the following orientation of A_5:

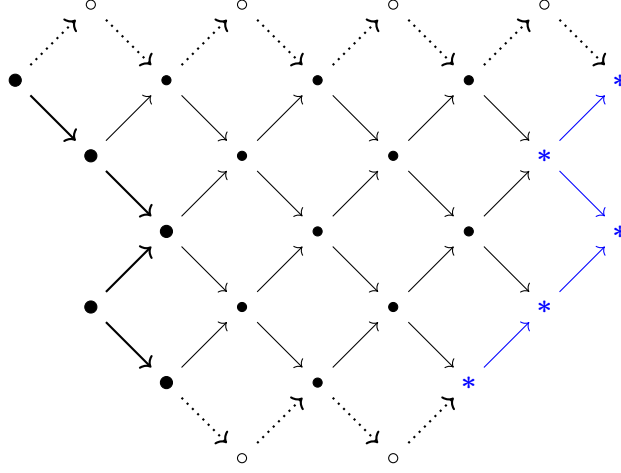$$1 \xleftarrow{\ \alpha\ } 2 \xleftarrow{\ \beta\ } 3 \xrightarrow{\ \gamma\ } 4 \xleftarrow{\ \delta\ } 5$$

FIGURE 1. Augmented Auslander–Reiten quiver for $1 \leftarrow 2 \leftarrow 3 \rightarrow 4 \leftarrow 5$.

where $\{3,5\}$ is the set of sources and $\{1,4\}$ consists of sinks in Q_0.

The algebra $\Lambda = kQ$ is finite dimensional and $B := \{e_1, e_2, e_3, e_4, e_5, \alpha, \beta, \gamma, \delta, \alpha\beta\}$ is a k-basis for Λ, where e_i denotes the trivial path at vertex i, for each $1 \leq i \leq 5$. Moreover, the multiplication in Λ is given by the composition of directed paths, whenever possible, and zero otherwise. For example $0 \neq \alpha\beta \in B$, but $\beta\alpha = 0$ and $\gamma\beta = 0$.

In this case, Γ_Λ is the subquiver of the quiver in Figure 1 determined by the solid vertices (those of the form $\bullet$) and the arrows between them. In particular, the leftmost copy of Q^{op} in Γ, which appears in bold, has the projective indecomposable modules $P_i := \Lambda e_i$ as its vertices, where $1 \leq i \leq 5$. Moreover, the additive subcategory $\mathcal{C}$ of $\mathcal{D}^b(kA_5)$ is generated by $\mathrm{Ind}(kA_5) \sqcup \{P_i[1] \mid 1 \leq i \leq 5\}$, where $P_i[1]$ is the shift of the projective module P_i in $\mathcal{D}^b(kA_5)$. The Auslander–Reiten quiver $\Gamma_\mathcal{C}$ is obtained by extending Γ_Λ, as further explained below.

In Figure 1, $\Gamma_\mathcal{C}$ is the quiver consisting of both solid and star vertices, together with all arrows between them. The vertices depicted by $\circ$ and the dotted arrows in Figure 1 are called *virtual*, for the reason we soon explain in the following.

Following the same notation from Example 2.1, if Q is a simply-laced Dynkin quiver and $\Lambda = kQ$, the leftmost copy of Q^{op} in Γ_Λ is called the *projective slice*, while the rightmost copy of Q^{op} in $\Gamma_\mathcal{C}$ is called the *shifted projective slice*. This is because for all $i \in Q_0$, the vertices of these copies of Q^{op} respectively correspond to the projective indecomposable modules $P_i = \Lambda e_i$ and their shifts $P_i[1]$ in $\mathcal{D}^b(\Lambda)$. For each P_i, the dimension vector $\underline{\dim}(P_i) \in \mathbb{Z}_{\geq 0}^{Q_0}$ is given as following: For any $j \in Q_0$, the jth coordinate of $\underline{\dim}(P_i)$ is the number of paths from i to j in Q. For each quiver Q treated in this work, and each pair of vertices i and j in Q, there is at most one directed path from i to j. Hence, for a fixed vertex i, the coordinates of the dimension vector $\underline{\dim}(P_i)$ in $Q_0^{\mathbb{Z}_{\geq 0}}$ always belong to $\{0,1\}$, whereas for an arbitrary quiver Q the coordinates of $\underline{\dim}(P_i)$ can be arbitrarily large. In particular, we always have the ith coordinate of $\underline{\dim}(P_i)$ is 1. Since the Auslander–Reiten theory induces mesh relations on $\Gamma_\mathcal{C}$, we can start from the projective slice and dimension vectors of projective modules and use the mesh relations to determine the dimension vector $\underline{\dim}(X)$, for each vertex X of $\Gamma_\mathcal{C}$.

It is well known that each vertex of $\Gamma_\mathcal{C}$ corresponds to a cluster variable of the cluster algebra associated to the simply-laced Dynkin quiver Q. Recall that two cluster variables are said to be *compatible* if there is some cluster which contains both of them. Consequently, two elements of $\Gamma_\mathcal{C}$ are called compatible if the corresponding cluster variables are such.

The case that generalizes to our continuous setting is when we take $\Lambda = kQ$ for Q a type A_n quiver (for $n \in \mathbb{Z}_{>1}$). Here, we consider $\mathcal{C} := \mathrm{add}\left(\mathrm{Ind}(\Lambda) \sqcup \{P_i[1] \mid i \in Q_0\}\right)$ as a full subcategory of

$\mathcal{D}^b(\Lambda)$, and we extend the Auslander–Reiten quiver $\Gamma_{\mathcal{C}}$ to a larger quiver $\widetilde{\Gamma}_A$ which plays a prominent role in our studies. For an inductive construction of $\widetilde{\Gamma}_A$ compatible with our continuous setting, we fix the following particular configuration of $\Gamma_{\mathcal{C}}$ as the initial step:

(0) Suppose $\Gamma_{\mathcal{C}}^0$ is a realization of $\Gamma_{\mathcal{C}}$ in $\mathbb{R}^2$ such that each arrow of $\Gamma_{\mathcal{C}}$ is of unit length which makes a ±45-degree angle with the horizon, and the Auslander–Reiten translation is a horizontal shift to the left.

It follows that any mesh relation in $\Gamma_{\mathcal{C}}^0$ forms a diamond or an isosceles triangle. Now that the initial step of our inductive construction is described, we extend $\Gamma_{\mathcal{C}}^0$ to the desired translation quiver $\widetilde{\Gamma}_\Lambda$. In particular, provided $\Gamma_{\mathcal{C}}^i$ is constructed, the next step is as follows:

$(i+1)$ In $\Gamma_{\mathcal{C}}^i$, if x is a leftmost vertex of odd degree, add an arrow δ_x of unit length outgoing from x and pointing to the East such that δ_x is orthogonal to all arrows of $\Gamma_{\mathcal{C}}^i$ connected to x. Denote the resulting quiver by $\Gamma_{\mathcal{C}}^{i+1}$.

We say δ_x is a *virtual arrow* of $\Gamma_{\mathcal{C}}^{i+1}$. Moreover, if $e(\delta_x)$ does not belong to $\Gamma_{\mathcal{C}}$, it is called a *virtual vertex* of $\Gamma_{\mathcal{C}}^{i+1}$. Consequently, $\Gamma_{\mathcal{C}}^{i+1}$ is said to be the *virtual extension* of $\Gamma_{\mathcal{C}}^i$ by δ_x. Finally, if $\Gamma_{\mathcal{C}}^{i+1}$ has no vertex of odd degree, set $\widetilde{\Gamma}_\Lambda := \Gamma_{\mathcal{C}}^{i+1}$ and call it the *augmented Auslander–Reiten quiver* of A. From the construction it is easy to see that there is a unique $i \in \mathbb{Z}$ with $\widetilde{\Gamma}_\Lambda = \Gamma_{\mathcal{C}}^{i+1}$.

We observe that $\widetilde{\Gamma}_\Lambda$ can be tiled by squares (symmetric diamonds) of the same size. Moreover, the projective slice (resp. the shifted projective slice) in $\Gamma_{\mathcal{C}}$ determines the leftmost (resp. rightmost) border of $\widetilde{\Gamma}_\Lambda$. All arrows in $\widetilde{\Gamma}_\Lambda \setminus \Gamma_{\mathcal{C}}$ are virtual, but there are virtual arrows whose endpoints belong to $\Gamma_{\mathcal{C}}$, meaning that $e(\delta_x)$ is not necessarily virtual. In fact, the top and bottom rows of $\widetilde{\Gamma}_\Lambda$ consists of virtual vertices and each virtual vertex of $\widetilde{\Gamma}_\Lambda$ belongs to exactly one of these two rows.

For instance, if Q is the quiver from Example 2.1, then in Figure 1 we can see the explicit construction of the augmented quiver $\widetilde{\Gamma}_\Lambda$ for $\Lambda = kQ$ via the above algorithms. In this case, we iterate the second step of the algorithm twelve times, each time creating a new virtual arrow shown by dotted arrows. The virtual vertices are depicted by circle (nonsolid) vertices, which form the top and bottom rows of vertices in $\widetilde{\Gamma}_\Lambda$.

Recall that each vertex of $\Gamma_{\mathcal{C}}$ comes with its own dimension vector. Further, to any virtual vertex v of $\widetilde{\Gamma}_\Lambda$ we associate the *virtual dimension vector*, being $\underline{\dim}(v) = (0, 0, \dots, 0) \in \mathbb{Z}^{Q_0}$. For each diamond tile $\Diamond$ in $\widetilde{\Gamma}_\Lambda$, let $L_\Diamond, R_\Diamond, U_\Diamond$, and $D_\Diamond$ respectively denote the left, right, top and bottom vertices of $\Diamond$. Then, thanks to the Auslander–Reiten translation in $\Gamma_{\mathcal{C}}$ and the induced mesh relations, the following identity holds for any diamond $\Diamond$ in $\widetilde{\Gamma}_\Lambda$:

$$\underline{\dim}(L_\Diamond) + \underline{\dim}(R_\Diamond) = \underline{\dim}(U_\Diamond) + \underline{\dim}(D_\Diamond).$$

From the above identity and a straightforward computation, one obtains similar equations for any rectangular area that entirely lies in the quiver $\widetilde{\Gamma}_\Lambda$. In particular, every such rectangle is tiled with the symmetric diamonds. Via the cancellations induced by these diamonds, an analogous equation holds for the rectangle, where $L_\Diamond, R_\Diamond, U_\Diamond$, and $D_\Diamond$ should be replaced by the appropriate corners of the rectangle.

Remark 2.2. The notion of compatibility of vertices in $\Gamma_{\mathcal{C}}$ also has a more homological incarnation and can be phrased in terms of extensions between the corresponding indecomposable objects in $\mathcal{C}$. In particular, as shown in [BM+06], two vertices v and w of $\Gamma_{\mathcal{C}}$, with the corresponding indecomposable objects M_v and M_w in $\mathcal{C}$, are compatible if and only if $\mathrm{Ext}^1_{\mathcal{C}}(M_v, M_w) = \mathrm{Ext}^1_{\mathcal{C}}(M_w, M_v) = 0$. As described above, we can view v and w as vertices of $\widetilde{\Gamma}_\Lambda$. This implies that M_v and M_w are incompatible if and only if there exists a rectangle that fully lies in $\widetilde{\Gamma}_\Lambda$ whose left and right corners are v and w (or w and v).

Suppose $\mathcal{I}^+$ denotes the set of vertices in $\widetilde{\Gamma}_\Lambda$ associated to $\mathrm{Ind}(\Lambda)$. The dimension vectors for all such vertices are nonzero with nonnegative coordinates. This justifies the choice of notation $\mathcal{I}^+$. From the construction of $\widetilde{\Gamma}_\Lambda$, it follows that any vertex in $\widetilde{\Gamma}_\Lambda$ which does not belong to $\mathcal{I}^+$ is either

a virtual vertex or it is associated to $P_i[1]$, for some $i \in Q_0$. In the former case the associated dimension vector is virtual and thus the zero vector, but in the latter case the associated dimension vector is nonzero with nonpositive coordinates. In particular, the set of vertices of $\widetilde{\Gamma}_\Lambda$ is a disjoint union of the form $\mathcal{I}^+ \sqcup \mathcal{I}^v \sqcup \mathcal{I}^{[1]}$, where $\mathcal{I}^v$ denotes the set of virtual vertices, and $\mathcal{I}^{[1]}$ consists of vertices associated to the shifted projective indecomposable objects $P_i[1]$ in $\mathcal{C}$, for all $i \in Q_0$. For the dimension vectors associated to the vertices of $\widetilde{\Gamma}_\Lambda$, we often refer to those corresponding to $\mathcal{I}^+$, $\mathcal{I}^v$ and $\mathcal{I}^{[1]}$ respectively as the positive, zero and negative dimension vectors.

In addition to the set of dimension vectors, the collection of g-vectors corresponding to the positive and negative vertices of $\widetilde{\Gamma}_\Lambda$ satisfy all the mesh relations. It is well known that the g-vectors of the indecomposable modules in the projective slice, being the leftmost boundary of $\widetilde{\Gamma}_\Lambda$, are given by the standard basis of $\mathbb{R}^{Q_0}$. Therefore, if to each vertex in $\mathcal{I}^v$ we again associate the zero vector, one can use the mesh relations in $\widetilde{\Gamma}_\Lambda$ to compute the g-vectors of all the remaining indecomposable modules in $\widetilde{\Gamma}_\Lambda$. This new set of vectors has played a pivotal role in the generalization of the setting and results in [AB+18]. This is because the physical phenomena studied by the authors resulted in a system of equations which could be realized as the deformed mesh relations that hold in the augmented quiver $\widetilde{\Gamma}_\Lambda$, where $\Lambda = kA_n$ for the linearly oriented quiver A_n. In particular, for each diamond $\Diamond$ in $\widetilde{\Gamma}_\Lambda$, the *deformed mesh relation* is of the form

$$(2) \qquad \underline{\dim}(L_\Diamond) + \underline{\dim}(R_\Diamond) = \underline{\dim}(U_\Diamond) + \underline{\dim}(D_\Diamond) + c_\Diamond$$

where $c_\Diamond$ is a nonnegative real value associated to $\Diamond$. We observe that for each rectangular area that entirely lies in the quiver $\widetilde{\Gamma}_\Lambda$, the deformed mesh relations imply a similar equation, while the constant $c_\Diamond$ must be replaced by the sum of all $c_\Diamond$'s for the diamonds $\Diamond$ that tile the rectangle. Furthermore, it is obvious that if $c_\Diamond = 0$, for every diamond $\Diamond$ in $\widetilde{\Gamma}_\Lambda$, we get the old system of equations induced by the mesh relations.

In retrospect, in [AB+18] the authors considered a polytopal realization of the space induced by the deformed mesh relations coming from $\widetilde{\Gamma}_\Lambda$, where $\Lambda = kA_n$ is given by a linearly oriented quiver A_n. This phenomenon was in fact viewed as a geometric description of the scattering amplitudes for bi-adjoint ϕ^3 scalar theory and plays an analogous role to that of a particular semialgebraic set in a Grassmannian, called the *amplituhedron*, which encodes scattering amplitudes for $\mathcal{N} = 4$ super Yang–Mills theory. Due to the analogous nature of the aforementioned polytopal realization in [AB+18], we adopt this terminology and henceforth refer to that as the amplituhedron for bi-adjoint ϕ^3 scalar theory. This realization of the amplituhedron was generalized to all simply-laced Dynkin quivers in terms of *generalized associahedra*—the polyhedra whose normal fan is given by the g-vectors (see [BD+18]). This last realization occurs in a kinematic space $\mathbb{V} = \prod_{\mathcal{I}} \mathbb{R}$, where $\mathcal{I} := \mathcal{I}^+ \sqcup \mathcal{I}^{[1]}$. Provided $\underline{c} = (c_\Diamond)_{\Diamond \in \widetilde{\Gamma}_\Lambda}$ is a collection of positive integers, in [BD+18] the authors consider an n-dimensional affine space $\mathbb{E}_{\underline{c}}$ inside $\mathbb{V}$ determined by $\underline{c}$. Here n is the number of vertices of the quiver. In fact, $\mathbb{E}_{\underline{c}}$ is induced by the $\underline{c}$-deformed mesh relations of the form

$$p_{i,j} + p_{i+1,j} = c_{ij} + \sum_{(i,j) \to (i',j') \to (i+1,j)} p_{i',j'},$$

where $p_{i,j}$ is the coordinate function on $\mathbb{V}$ indexed by (i,j) in $\mathcal{I}$. Let $\mathbb{U}_{\underline{c}}$ denote the intersection of the positive orthant in $\mathbb{V}$ with $\mathbb{E}_{\underline{c}}$. In [BD+18], the authors further use these $p_{i,j}$ and the g-vectors to define a projection of $\mathbb{V}$ onto the n-dimensional real space. In particular, they consider $\mathbb{A}_{\underline{c}} = \pi(\mathbb{U}_{\underline{c}})$, where $\pi : \mathbb{V} \to \mathbb{R}^n$, whose kth coordinate is given by p_{ij} for $(i,j) \in \mathcal{I}^{[1]}$ such that $g(i,j) = -e_k$. Here e_k denotes the kth standard bases of $\mathbb{R}^n$. More specifically, they show the following theorem. For further details, see [BD+18].

Theorem 2.3. (1) *Each facet of $\mathbb{U}_{\underline{c}}$ is defined by the vanishing of exactly one coordinate of $\mathbb{V}$. Moreover, the vertices of $\mathbb{U}_{\underline{c}}$ correspond to clusters.*

(2) *The faces of $\mathbb{A}_{\underline{c}}$ correspond bijectively to compatible sets in $\mathcal{I}$.*
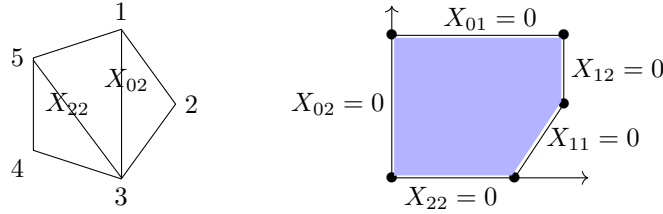
FIGURE 2. For $n = 2$, the kinematic space is $\mathbb{R}^5$. The associahedron is realized inside a 2-dimensional affine plane determined by a system of inequalities in terms of the mesh relations. The facets correspond to diagonals of the pentagon, and vertices correspond to its triangulations.

In our treatment of the more general setting which will be discussed in the following sections, $\underline{c}$ need not be an integer vector, meaning that for each diamond $\lozenge$ in $\widetilde{\Gamma}_\Lambda$, we only assume $c_\lozenge$ is a nonnegative real value.

Remark 2.4. Since the dimensions of the kinematic space $\mathbb{V} = \prod_{\mathcal{I}} R$ and the subspace $\mathbb{U}_{\underline{c}}$ grow fast, visualization of the amplituhedron $\mathbb{A}_{\underline{c}}$ is possible only for small Dynkin quivers. In Figure 2 we consider the case $Q = A_2$. In particular, $\mathbb{A}_{\underline{c}}$ is the ordinary associahedron for $\underline{c} \in \mathbb{R}^3_{>0}$, where the X_{ij} denote the (affine-)linear forms on $\mathbb{R}^{\mathcal{I}}$, indexed by the vertices of $\Gamma_{\mathcal{C}}$, as labelled below:
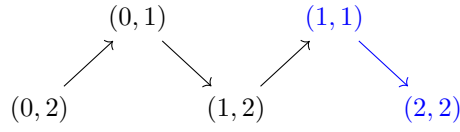
We note that each choice of $\underline{c} \in \mathbb{R}^3_{\geq 0}$ yields different deformed mesh relations and the corresponding system of equations gives rise to different generalized associahedra $\mathbb{A}_{\underline{c}}$ in $\mathbb{R}^2$. Regardless of the choice of $\underline{c}$, we remark that all the generalized associahedra $\mathbb{A}_{\underline{c}}$ have two pairs of parallel facets, which are $\{X_{01}, X_{22}\}$ and $\{X_{02}, X_{12}\}$.

2.2. **The category $\mathcal{D}$.** We now introduce a continuous version of $\mathcal{D}^b(A_n)$, which we denote by $\mathcal{D}$. This category is triangulated equivalent to the category $\mathcal{D}_\pi$ defined by Igusa and Todorov [IT15]. Moreover, these categories are isomorphic: they are equivalent and their objects (not just isomorphism classes) are in bijection.

2.2.1. *Objects and morphisms.* The indecomposable objects of $\mathcal{D}$ are the points in the set $\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$, and each object in $\mathcal{D}$ is a finite (possibly empty) direct sum of indecomposable objects.
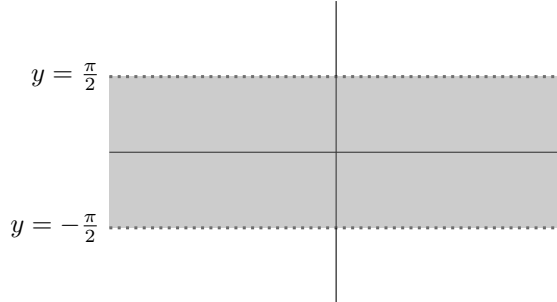
For each point (x, y), define the set $H(x, y)$ in the following way. First, consider the rectangle determined by the points

$$(x, y), \quad \left(x + \frac{\pi}{2} - y, \frac{\pi}{2}\right), \quad \left(x + \frac{\pi}{2} + y, -\frac{\pi}{2}\right), \quad \text{and} \quad (x + \pi, -y).$$
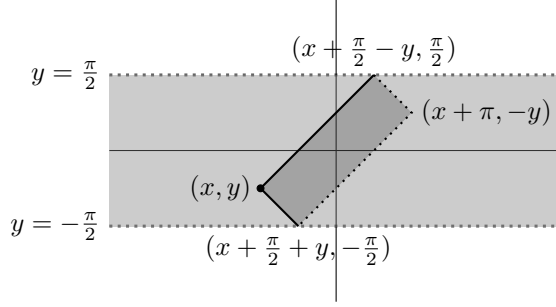
FIGURE 3. Morphisms between indecomposable objects in $\mathcal{D}$.

The set $H(x, y)$ is the interior of this rectangle together with the left boundary, but without the points $(x + \frac{\pi}{2} - y, \frac{\pi}{2})$ and $(x + \frac{\pi}{2} + y, -\frac{\pi}{2})$. One may also consider $H(x, y)$ as being defined by the beams emitted from (x, y) with slopes ± 1. See Figure 3 for an illustration of the set $H(x, y)$.

We define Hom between indecomposable objects as follows:

$$\mathrm{Hom}((x, y), (x', y')) = \begin{cases} k & \text{if } (x', y') \in H(x, y) \\ 0 & \text{otherwise.} \end{cases}$$

When $(x, y) = (x', y')$, the identity morphism on (x, y) is $1 \in k$. The composition of morphisms $f : (x_1, y_1) \to (x_2, y_2)$ and $g : (x_2, y_2) \to (x_3, y_3)$ is given by multiplication of f and g inside k whenever $\mathrm{Hom}((x_1, y_1), (x_3, y_3))$ is not 0. Hom sets and composition of morphisms for arbitrary objects is given by extending the structure bilinearly.

2.2.2. *Triangulated structure.* Let (x, y) be an indecomposable object in $\mathcal{D}$. We define the shift of (x, y) by

$$(x, y)[1] := (x + \pi, -y).$$

The shift of a sum of indecomposables is defined to be the sum of the shift of each of the indecomposables.

The minimal distinguished triangles in $\mathcal{D}$ are of the form

$$(x, y) \longrightarrow E \longrightarrow (x', y') \longrightarrow (x + \pi, -y),$$

where the case $E = (x, y) \oplus (x', y')$ is called a *trivial* triangle.

Among these minimal distinguished triangles, we are particularly interested in the nontrivial ones. This is because, as in Definition 2.5 and Section 5, we use them to discuss the notions of tilting rectangles (Definition 2.5) and compatibility (Definition 5.2). Moreover, the rest of the distinguished triangles in $\mathcal{D}$ can be constructed from this collection of minimal distinguished triangles (see [IT15] for details).

The other possibility occurs when (x', y') is an element of $\overline{H(x, y)} \cap \left(\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2}) \right)$, where the overline denotes the closure of $H(x, y)$ in $\mathbb{R}^2$ with the standard Euclidean topology. In this case, consider the following two points:

$$T = \left(\frac{x + x' - y + y'}{2}, \frac{-x + x' + y' + y}{2} \right)$$

$$B = \left(\frac{x + x' + y - y'}{2}, \frac{x - x' + y' + y}{2} \right).$$

If the y-coordinate of T is $\frac{\pi}{2}$ then we set $T = 0$ in $\mathcal{D}$, and similarly for B if the y-coordinate of B is $-\frac{\pi}{2}$. Thus, each of T and B is an indecomposable object or 0 in $\mathcal{D}$.

These can be seen in Figure 4 as the top and bottom points of the rectangle whose respective left and right endpoints are (x, y) and (x', y'). If $T = B = 0$ in $\mathcal{D}$ then $(x', y') = (x + \pi, -y)$, and we
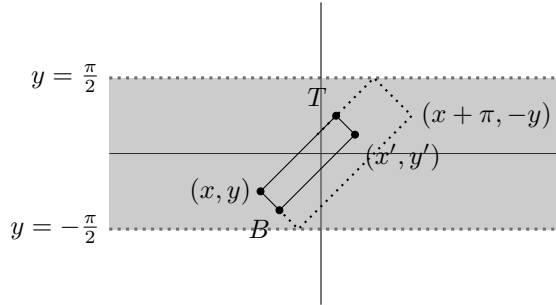

FIGURE 4. An example of a tilting rectangle corresponding to the distinguished triangle $(x, y) \to T \oplus B \to (x', y') \to (x + \pi, -y)$.

have the distinguished triangle

$$(x, y) \longrightarrow 0 \longrightarrow (x + \pi, -y) \xrightarrow{\ \cong\ } (x + \pi, -y).$$

The rectangles we obtain from such distinguished triangles will play a key role in our construction of the continuous associahedron.

Definition 2.5. Let $X = (x, y)$ be a point in the strip $\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$. Let a and b be positive real numbers such that

$$0 < a \le \frac{\pi}{2} - y$$
$$0 < b \le y + \frac{\pi}{2}.$$

Let Y, Z, and W be the points

$$Y = (x + a, y + a)$$
$$Z = (x + b, y - b)$$
$$W = (x + a + b, y + a - b).$$

We call the rectangle $XYWZ$ a *tilting rectangle*.

We see an example of a tilting rectangle in Figure 4.

Proposition 2.6. *There is a bijection between tilting rectangles in $\mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and distinguished triangles in $\mathcal{D}$ whose first and third terms are both nonzero and indecomposable.*

Proof. This follows from Igusa and Todorov's construction [IT15, Section 2]. $\square$

This proposition asserts that the set $\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ acts like the Auslander–Reiten quiver of $\mathcal{D}$ and $\mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ acts like the augmented Auslander–Reiten quiver of $\mathcal{D}$. In [IT15] the authors show that this, along with a choice of which triangles are distinguished, are sufficient to yield a triangulated structure on $\mathcal{D}$. This choice does not affect our constructions.

2.3. **Continuous mesh relations.** In the finite and discrete setting, the mesh relations are induced by the almost split triangles in $\mathcal{D}^b(k\mathbb{A}_n)$. Recall that almost split triangles are not trivial. However, in $\mathcal{D}$, the only irreducible morphisms are isomorphisms. Thus, we cannot have any almost split triangles. Nevertheless, a continuous version of the mesh relations hold instead.

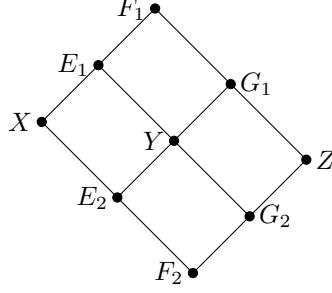From now on, we use capital letters to denote indecomposable objects in $\mathcal{D}$.

FIGURE 5. A schematic of distinguished triangles in $\mathcal{D}$ and the corresponding rectangles in $\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$.

Definition 2.7. Let $\mathbb{V}$ be an arbitrary real vector space and $\Phi : \mathrm{Ind}(\mathcal{D}) \sqcup \{0\} \to \mathbb{V}$ a function satisfying $\Phi(0) = 0$. We extend Φ to all objects of $\mathcal{D}$ by defining its value on a direct sum to be the sum of the values of Φ on each indecomposable summand.

We say Φ *satisfies the continuous mesh relations* provided that for every distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1],$$

where X and Z are indecomposable, the following equation holds:

$$\Phi(X) + \Phi(Z) = \Phi(Y).$$

Two examples of such functions are $g_{\mathcal{Z}}$-vectors (Section 4.2) and dimension vectors (Section 4.3). We now set the context for an important property of functions satisfying continuous mesh relations, which is stated in Proposition 2.8. Suppose there are nontrivial distinguished triangles

$$(1) \qquad X \longrightarrow E_1 \oplus E_2 \longrightarrow Y \longrightarrow X[1]$$

$$(2) \qquad Y \longrightarrow G_1 \oplus G_2 \longrightarrow Z \longrightarrow Y[1]$$

$$(3) \qquad X \longrightarrow F_1 \oplus F_2 \longrightarrow Z \longrightarrow X[1].$$

Without loss of generality, we assume that E_1, F_1, and G_1 are top points of the tilting rectangles in $\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ obtained from the given distinguished triangles in $\mathcal{D}$.

The relationship between the geometry in $\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ and the distinguished triangles in $\mathcal{D}$ yields the schematic of the points in $\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ in Figure 5. It can be seen from the schematic that we have two more distinguished triangles:

$$(4) \qquad E_1 \longrightarrow F_1 \oplus Y \longrightarrow G_1 \longrightarrow E_1[1]$$

$$(5) \qquad E_2 \longrightarrow Y \oplus F_2 \longrightarrow G_2 \longrightarrow E_2[1].$$

We see that distinguished triangles (1), (3) yield distinguished triangles (2), (4), and (5) both algebraically in $\mathcal{D}$ and geometrically in $\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$.

The following proposition highlights how the continuous mesh relations behave similarly to the discrete version.

Proposition 2.8. *Assume Φ satisfies the continuous mesh relations. Let X, E_1, E_2, F_1, and F_2 be indecomposables in $\mathrm{Ind}(\mathcal{D})$. Let $X \to E_1$ and $X \to E_2$ be morphisms, the slope from X to E_1 be 1, and the slope from X to E_2 be -1. Also let $E_1 \to F_1$ and $E_2 \to F_2$ be morphisms, the slope from X to F_1 be 1, and the slope from X to F_2 be -1.*

Then one obtains the distinguished triangles (1)–(5) above. Furthermore, the values $\Phi(X)$, $\Phi(E_1)$, $\Phi(E_2)$, $\Phi(F_1)$, and $\Phi(F_2)$ determine $\Phi(Y)$, $\Phi(G_1)$, $\Phi(G_2)$, and $\Phi(Z)$.

Proof. The application of the octahedral axiom yields the distinguished triangles. We see that
$$\Phi(Y) = \Phi(E_1) + \Phi(E_2) - \Phi(X).$$
Then we may replace $\Phi(Y)$ in the equations for $\Phi(G_1)$ and $\Phi(G_2)$. Finally we replace $\Phi(Y)$, $\Phi(G_1)$, and $\Phi(G_2)$ in the equation for $\Phi(Z)$. □

Note that, in the proposition above, if any of E_1, E_2, F_1, and F_2 are the 0 object then we may choose points on the boundary of $\mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that the assertions hold.

3. Continuous deformed mesh relations

The goal of this section is to introduce the continuous version of the deformed mesh relations (2) from Section 2.1. For that we first develop a technique that we call *quilting*, which does not depend on knitting.

3.1. **Quilting.**

3.1.1. *Patches and quilting.* Notice that in Section 2.1 the projective slice forms a zigzag shape. To introduce a continuous version of the projective slice, we begin with the following definition.

Definition 3.1. Let $L = \{\ell_1, \ldots, \ell_n\}$ be a set of distinct line segments in $\mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ which satisfy the following conditions.

- The slope of ℓ_1 is ± 1, and its top point is on the line $y = \frac{\pi}{2}$.
- For $1 \leq i < n$, the segment ℓ_{i+1} is below ℓ_i and they intersect at a right angle.
- The bottom point of ℓ_n lies on the line $y = -\frac{\pi}{2}$.

Then we set
$$\mathcal{Z} = \left(\bigcup_{i=1}^{n} \ell_i \right) \cap \left(\mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \right),$$
and call it a *zigzag with n line segments*. We consider $\mathcal{Z}$ as a full subcategory of $\mathcal{D}$.

For each $1 \leq i < n$, the second condition implies ℓ_{i+1} and ℓ_i have exactly one point $P_i = (a_i, b_i)$ in common and $y < b_i$, for any other point (x, y) on ℓ_{i+1}. If the slope of ℓ_i is 1 (respectively, -1), this unique point P_i is called a *left vertex* (respectively, *right vertex*) of $\mathcal{Z}$.

In Definition 3.1, the top point of ℓ_1 and the bottom point of ℓ_n do not belong to $\mathcal{Z}$, and all points in $\mathcal{Z}$ are indecomposable objects in the category $\mathcal{D}$.

Next we define the two types of patches. We will save visual depictions until we define quilting.

Definition 3.2. Let $\mathcal{Z}$ be a zigzag with n line segments.

- Let ℓ_i and ℓ_{i+1} be line segments of $\mathcal{Z}$ that share a left vertex. Let the top point of ℓ_i be $(y_1, y_2) = Y$ and the bottom point of ℓ_{i+1} be $(z_1, z_2) = Z$. Let the shared left vertex be $(x, y) = X$, and let
$$W = (y_1 + z_1 - x_1, y_2 + z_2 - x_2).$$
Then $XYWZ$ is a rectangle in $\mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ whose sides have slopes ± 1. By a *rectangular patch*, we mean the part of $XYWZ$ in $\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ together with the region it bounds.
- If ℓ_i is a line segment that does not share a left vertex with another line segment, then $i = 1$ or $i = n$. Let Y be the right vertex on ℓ_i and X the point in $\mathbb{R} \times \{-\frac{\pi}{2}, \frac{\pi}{2}\}$ on ℓ_i. We reflect ℓ_i about the vertical line through Y. Then there is a point Z in both $\mathbb{R} \times \{-\frac{\pi}{2}, \frac{\pi}{2}\}$ and the reflection of ℓ_i. This defines a triangle XYZ in $\mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. By a *triangular patch*, we mean the part of XYZ in $\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ together with the region it bounds.

Henceforth, by a *patch* we mean a rectangular patch or a triangular patch.

Note that patches are always to the right of the corresponding zigzag.

Definition 3.3 (Quilting)**.** For a zigzag $\mathcal{Z}$, we define *quilting* $\mathfrak{Q}(\mathcal{Z})$ of $\mathcal{Z}$ via a process on all patches induced by $\mathcal{Z}$.

First, let $XYWZ$ be a rectangular patch. By $\mathfrak{Q}(\ell_{i+1})$ we denote the top right side of $XYWZ$, which is a translation of ℓ_{i+1}. Furthermore, by $\mathfrak{Q}(\ell_i)$ we denote the bottom right side of $XYWZ$, which is a translation of ℓ_i. Note that left vertices of $\mathfrak{Q}(\ell_i) \cup \mathfrak{Q}(\ell_{i+1})$ are the right vertices of $\ell_i \cup \ell_{i+1}$. Also note that $\mathfrak{Q}(\ell_{i+1})$ is now above $\mathfrak{Q}(\ell_i)$.
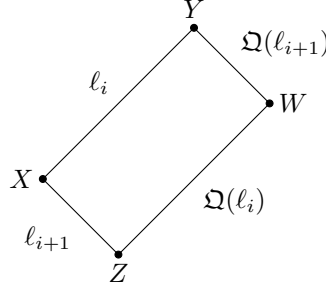
Now let XYZ be a triangular patch. We define $\mathfrak{Q}(\ell_i)$ to be the reflection of ℓ_i across the vertical line through Y.
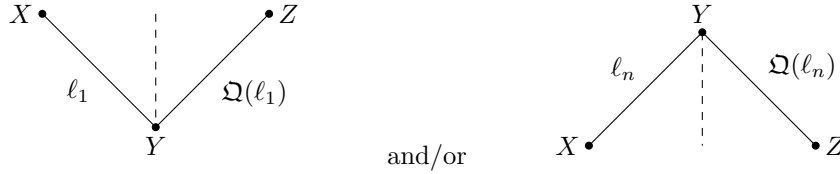
We define $\mathfrak{Q}(\mathcal{Z}) := \left(\bigcup_i \mathfrak{Q}(\ell_i) \right) \cap \left(\mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \right)$. Notice again that the left vertex of $\mathfrak{Q}(\ell_i)$ is the right vertex of ℓ_i. In the top-to-bottom ordering of the line segments of $\mathfrak{Q}(\mathcal{Z})$, we consider these reflections to be fixed when compared to $\mathcal{Z}$.

Remark 3.4. It follows directly from the definition that $\mathfrak{Q}(\mathcal{Z})$ is unique. Moreover, in $\mathbb{R} \times \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$, a rectangular patch is a tilting rectangle.

3.1.2. *Requisite combinatorics.* In this section, we collect the combinatorial tools needed for our proofs about quilting and continuous deformed mesh relations. We treat permutations as functions, and therefore if α and β are permutations of a set X, by $\beta \circ \alpha$ we denote the permutation obtained by first performing α and then performing β. If $[\mathbf{n}, <]$ is an ordered set with n elements, say $\{x_1, \ldots, x_n\}$, by s_i we denote the simple permutation which swaps the position of x_i and x_{i+1}. In particular, $s_i([\mathbf{n}, <]) = \{x_1, \ldots, x_{i-1},\ x_{i+1}, x_i,\ x_{i+2}, \ldots, x_n\}$.

For any $m \in \mathbb{R}$, let $\lceil m \rceil := \min\{a \in \mathbb{Z} \mid m \le a\}$ and $\lfloor m \rfloor := \max\{b \in \mathbb{Z} \mid b \le m\}$.

Definition 3.5. We consider two particular permutations on the ordered set $[\mathbf{n}, <]$:

    (1) Let $\pi_1 := s_{2\lfloor \frac{n}{2} \rfloor - 1} \circ \cdots \circ s_3 \circ s_1$.
    (2) Let $\pi_2 := s_{2\lceil \frac{n-1}{2} \rceil} \circ \cdots \circ s_4 \circ s_2$.

One may think of π_1 as a permutation that moves elements in odd positions forward and elements in even positions backwards, where the indices permit.

Example 3.6. Consider the sets $\{A, B, C, D, E\}$ and $\{A, B, C, D, E, F\}$. We can visualize $\pi_2 \circ \pi_1$ on the left and right, respectively.
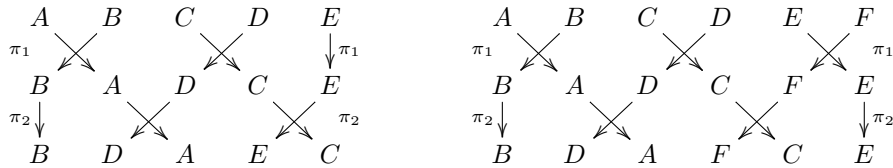
Observe that the permutations behave slightly differently depending on whether or not we are working with an even number of elements or an odd number of elements. For a permutation α, by $\text{Fix}(\alpha)$ we denote the set of elements fixed under α.

Proposition 3.7. *For* $[\mathbf{n}, <] = \{x_1, \ldots, x_n\}$, *we have the following properties.*

- *Both π_1 and π_2 are involutions on $[\mathbf{n}, <]$.*
- *If n is odd, then $\text{Fix}(\pi_1) = \{x_n\}$ and $\text{Fix}(\pi_2) = \{x_1\}$.*
- *If n is even, then $\text{Fix}(\pi_1) = \emptyset$ and $\text{Fix}(\pi_2) = \{x_1, x_n\}$.*
- $\text{Fix}(\pi_1) \cap \text{Fix}(\pi_2) = \emptyset$.

Proof. The statements follow directly from Definition 3.5. $\square$

Definition 3.8. Let $[\mathbf{n}, <]$ be an ordered set with n elements. For an even positive integer j, we define

- the *odd jth composition* to be $\pi_o^j := (\pi_2 \circ \pi_1)^{\frac{j}{2}}$,
- and the *even jth composition* to be $\pi_e^j := (\pi_1 \circ \pi_2)^{\frac{j}{2}}$.

For an odd positive integer j, we define

- the *odd jth composition* to be $\pi_o^j := \pi_1 \circ (\pi_2 \circ \pi_1)^{\frac{j-1}{2}}$,
- and the *even jth composition* to be $\pi_e^j := \pi_2 \circ (\pi_1 \circ \pi_2)^{\frac{j-1}{2}}$.

The naming convention above was chosen so that the odd compositions start with π_1 and the even compositions start with π_2. In essence, j counts the number of alternating compositions of π_1 and π_2 that we are performing.

Proposition 3.9. *Let* $[\mathbf{n}, <] = \{x_1, \ldots, x_n\}$. *For* $1 \leq i < n$ *and* $j = n - i$, *we have*

- *if i is odd, then $\pi_o^j(x_i) = x_n$ and $\pi_o^{j+1}(x_i) = x_n$.*
- *if i is even, then $\pi_e^j(x_i) = x_n$ and $\pi_e^{j+1}(x_i) = x_n$.*

Proof. The proof is by induction on j. First suppose $j = 1$; we use Proposition 3.7. If i is odd, then n is even. So $\pi_1(x_i) = x_n$ and $\pi_2(x_n) = x_n$. If instead i is even, then n is odd. In this case, $\pi_2(x_i) = x_n$ and $\pi_1(x_n) = x_n$.

Assume that the assertion holds for all positive integers less than or equal to ℓ, and let $j = \ell + 1$. Suppose i is odd. Then $\pi_1(x_i) = x_{i+1}$. We know $n - (i + 1) = \ell$ and the proposition holds for ℓ.

If we perform $\pi_e^\ell \circ \pi_1$ then we have performed π_o^j, which sends x_i to x_n. If we perform $\pi_e^{\ell+1} \circ \pi_1$ we have performed π_o^{j+1}, which sends x_i to x_n. If i is even, we can perform a similar argument. $\square$

Proposition 3.10. *Let* $[\mathbf{n}, <] = \{x_1, \ldots, x_n\}$. *For* $1 < i \leq n$ *and* $j = i - 1$, *we have*

- *if i is odd, then $\pi_e^j(x_i) = x_1$ and $\pi_e^{j+1}(x_i) = x_1$.*
- *if i is even, then $\pi_o^j(x_i) = x_1$ and $\pi_o^{j+1}(x_i) = x_1$.*

Proof. The statement and the proof are symmetric to Proposition 3.9 and its proof. $\square$

Lemma 3.11. *Let* $[\mathbf{n}, <]$ *be an ordered set with n elements. The following hold.*

- π_o^n *and* π_e^n *both reverse the order of the element in $[\mathbf{n}, <]$.*
- *If $x \in [\mathbf{n}, <]$, there exist unique $0 \leq i, j < n$ such that $\pi_o^i(x) = \pi_o^{i+1}(x)$ and $\pi_e^j(x) = \pi_e^{j+1}(x)$.*

Proof. We prove the statement by showing that the ith element is sent to the $((n+1) - i)$th element. First suppose i is odd. Then, by Proposition 3.9, both π_o^{n-i} and π_o^{n+1-i} send i to the nth element.

If n is odd π_e^{i-1} sends the nth element to the $((n+1) - i)$th element. In this case, $((n+1) - i)$ is odd; so $\pi_e^{i-1} \circ \pi_o^{n+1-i} = \pi_o^n$.

If n is even then π_o^{i-1} sends the nth element to the $((n+1)-i)$th element. In this case, $((n+1)-i)$ is even; so $\pi_o^{i-1} \circ \pi_o^{n+1-i} = \pi_o^n$. The case when i is odd and we start with the $((n+1) - i)$th even composition is similar.

Now suppose i is odd and we first perform π_e^i. By Proposition 3.10, π_e^i and π_e^{i-1} send the ith element to the first position. Thus, $\pi_e^i = \pi_2 \circ \pi_e^{i-1}$. We know the odd $(n - i)$th composition will

send the first element to the $((n + 1) - i)$th position. We also see that $\pi_o^{n-1} \circ \pi_e^i = \pi_e^n$. The case when i is even is similar. This concludes the proof. □

3.1.3. *Quilting.* For a zigzag $\mathcal{Z}$ with n line segments, we show in Theorem 3.16 below that performing quilting n times on $\mathcal{Z}$ yields $\mathcal{Z}[1]$.

Proposition 3.12. *Let $\mathcal{Z}$ be a zigzag with the set of line segments $L := \{\ell_1, \ldots, \ell_n\}$. Then $\mathfrak{Q}(\mathcal{Z})$ is a zigzag, and the order on $\{\mathfrak{Q}(\ell_i) \mid \ell_i \in L\}$ is $\pi_1(L)$ if the slope of ℓ_1 is $+1$, and $\pi_2(L)$ otherwise.*

Proof. By Definition 3.3, $\mathfrak{Q}(\mathcal{Z})$ is indeed a zigzag: all line segments have alternating slopes ± 1 and the top and bottom points respectively belong to the lines $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$. To show the assertion, we treat two cases based on the slope of ℓ_1.

We consider the case where the slope of ℓ_1 is $+1$ (the case where the slope of ℓ_1 is -1 is similar). In this case, the slope of all odd line segments are $+1$. Hence, the slope of all even line segments are -1. Thus, for each pair ℓ_i and ℓ_{i+1} that share a left vertex, i must be odd. So, $\mathfrak{Q}(\ell_{i+1})$ is above $\mathfrak{Q}(\ell_i)$. If $n > 1$ is odd, then, by Definition 3.3, $\mathfrak{Q}(\ell_n)$ is a reflection. If n is even there are no reflections. Thus, the top-to-bottom order of the line segments of $\mathfrak{Q}(\mathcal{Z})$ is given by $\pi_1(L)$. □

Proposition 3.13. *For a zigzag $\mathcal{Z}$, there exists a unique zigzag $\mathcal{Z}'$ such that $\mathfrak{Q}(\mathcal{Z}') = \mathcal{Z}$.*

Proof. One may check that the symmetric construction to Definition 3.3 also yields a zigzag $\mathcal{Z}'$. It follows directly that $\mathfrak{Q}(\mathcal{Z}') = \mathcal{Z}$ and is unique. □

Propositions 3.12 and 3.13 justify the following definition and notation.

Notation 3.14. Let $\mathcal{Z}$ be a zigzag. By $\mathfrak{Q}^{-1}(\mathcal{Z})$ we denote the zigzag $\mathcal{Z}'$ such that $\mathfrak{Q}(\mathcal{Z}') = \mathcal{Z}$. Furthermore, for $i \in \mathbb{Z}$, we set

- $\mathfrak{Q}^i(\mathcal{Z}) = \overbrace{\mathfrak{Q}(\mathfrak{Q}(\cdots(\mathfrak{Q}(\mathcal{Z}))\cdots))}^{i \text{ times}}$ if $i > 0$,
- $\mathfrak{Q}^i(\mathcal{Z}) = \mathcal{Z}$ if $i = 0$, and
- $\mathfrak{Q}^i(\mathcal{Z}) = \underbrace{\mathfrak{Q}^{-1}(\mathfrak{Q}^{-1}(\cdots(\mathfrak{Q}^{-1}(\mathcal{Z}))\cdots))}_{i \text{ times}}$ if $i < 0$.

Lemma 3.15. *Let $\mathcal{Z}$ be a zigzag and L the set of its n line segments. The line segments of the zigzag $\mathfrak{Q}^n(\mathcal{Z})$ are the translations of the reflections of those in $\mathcal{Z}$, in the reverse order from top-to-bottom.*

Proof. Notice that for the line segments ℓ_1 and ℓ_n of a zigzag, reflection implies $\mathfrak{Q}(\ell_1)$ and $\mathfrak{Q}(\ell_n)$ are respectively the first and last line segments of $\mathfrak{Q}(\mathcal{Z})$. If ℓ_1 in $\mathcal{Z}$ has slope $+1$ then $\mathfrak{Q}(\ell_2)$, which is the top of $\mathfrak{Q}(\mathcal{Z})$, has slope -1. If ℓ_1 in $\mathcal{Z}$ has slope -1 then $\mathfrak{Q}(\ell_1)$ has slope $+1$. Thus, the order of the line segments in $\mathfrak{Q}^n(\mathcal{Z})$ is given either by $\pi_o^n(L)$ or $\pi_e^n(L)$. In either case, by Lemma 3.11 the line segments of $\mathfrak{Q}(\mathcal{Z})$ are as described. □

We now prove the main result of the section. In Figure 6, we see an example of Theorem 3.16.

Theorem 3.16. *Let $\mathcal{Z}$ be a zigzag in $\mathcal{D}$ with n line segments. Then, $\mathfrak{Q}^n(\mathcal{Z}) = \mathcal{Z}[1]$ and $\mathfrak{Q}^{-n}(\mathcal{Z}) = \mathcal{Z}[-1]$.*

Proof. If $\mathcal{Z}$ has one line segment the theorem is immediate. Assume $\mathcal{Z}$ has at least two line segments. By Lemma 3.15, $\mathfrak{Q}^n(\mathcal{Z})$ and $\mathcal{Z}[1]$ have the same shape; i.e., they are translations of each other. Similarly, $\mathfrak{Q}^{-n}(\mathcal{Z})$ is a translation of $\mathcal{Z}[-1]$.

We show the top points of $\mathfrak{Q}^n(\mathcal{Z})$ and $\mathcal{Z}[1]$ are the same, which implies that the translation is the identity and $\mathfrak{Q}^n(\mathcal{Z})$ and $\mathcal{Z}[1]$ are the same. The symmetric argument shows $\mathfrak{Q}^{-n}(\mathcal{Z}) = \mathcal{Z}[-1]$.

We assume the slope of ℓ_1 is -1, as the proof when ℓ_1 has slope $+1$ is similar.

Let m be the largest odd number inclusively between 1 and n. We first show that the top point of $\mathfrak{Q}^n(\mathcal{Z})$ is the same as the top point of $\mathfrak{Q}^m(\ell_m)$. Then, by combining Propositions 3.9 and 3.10 with Lemma 3.11, we conclude that the top line segment of each zigzag obtained from $\mathcal{Z}$ is among

$$\mathfrak{Q}^0(\ell_1), \mathfrak{Q}(\ell_1), \mathfrak{Q}^2(\ell_3), \mathfrak{Q}^3(\ell_3), \ldots, \mathfrak{Q}^{i-1}(\ell_i), \mathfrak{Q}^i(\ell_i), \ldots$$
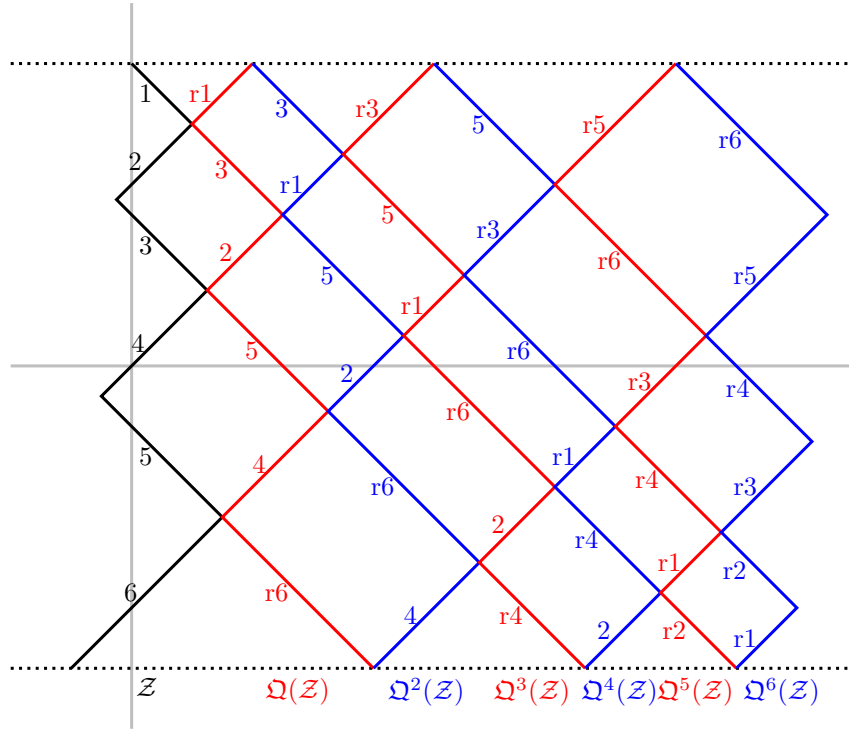
FIGURE 6. On a zigzag $\mathcal{Z}$ with 6 line segments, performing quilting 6 times results in $\mathcal{Z}[1]$. Here 'r' indicates the reflection of the line segment as in Definition 3.3.

where i is odd.

By Lemma 3.15, note that the top line segment of $\mathfrak{Q}^n(\mathcal{Z})$ is $\mathfrak{Q}^n(\ell_n)$. If n is odd this line segment has slope $+1$ since it was reflected exactly once. If n is even this line segment has slope -1 and so shares its top point with $\mathfrak{Q}^{n-1}(\ell_{n-1})$. Thus, the top point of $\mathfrak{Q}^n(\mathcal{Z})$ is the top point of $\mathfrak{Q}^m(\ell_m)$. We have also shown that the top point of $\mathfrak{Q}^i(\mathcal{Z})$ is the top point of $\mathfrak{Q}^{i+1}(\mathcal{Z})$ when $i < n$ is odd.

For each line segment ℓ_i, let x_{b_i} and x_{t_i} be respectively the x-coordinates of the bottom point and top point of ℓ_i. Let $h_i = |x_{t_i} - x_{b_i}|$. Now we show that the distance between the top point of $\mathcal{Z}$ and the top point of $\mathfrak{Q}^n(\mathcal{Z})$ is

$$2 \left(\sum_{\text{odd } 1 \leq i \leq n} h_i \right).$$

We see the distance between the top point of $\mathcal{Z}$ and the top point of $\mathfrak{Q}^1(\mathcal{Z})$ is $2h_1$. The distance between the top point of $\mathfrak{Q}^2(\mathcal{Z})$ and $\mathfrak{Q}^3(\mathcal{Z})$ is $2h_3$, and so on.

Let x_b (respectively, x_t) denote the x-coordinate of the bottom point (respectively, the top point) of $\mathcal{Z}$. Observe that

$$x_b - x_t = \left(\sum_{\text{odd } 1 \leq i \leq n} h_i \right) - \left(\sum_{\text{even } 1 \leq i \leq n} h_i \right).$$

Denote by x_q the x-coordinate of the top point of $\mathfrak{Q}^n(\mathcal{Z})$. From the previous equation it follows that

$$x_q - x_b = 2\left(\sum_{\text{odd } 1 \leq i \leq n} h_i\right) - \left(\sum_{\text{odd } 1 \leq i \leq n} h_i\right) + \left(\sum_{\text{even } 1 \leq i \leq n} h_i\right)$$

$$= \sum_{i=1}^{n} h_i = \pi.$$

This implies that the top point of $\mathfrak{Q}^n(\mathcal{Z})$ is $(x_b + \pi, \frac{\pi}{2})$, which is the top point of $\mathcal{Z}[1]$. $\qquad\square$

3.2. **Deformed mesh relations.** In this section, we define the continuous analog of the deformed mesh relations (2) in Section 2.1.

3.2.1. *Permissible functions.* We define the continuous analog of the $\underline{c}$ values in Section 2.1. In practice we want to be able to integrate such a $\underline{c}$ function over any tilting rectangle (Definition 2.5) in the closed strip $\mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. This will allow us to define what it means for a function $\Phi : \text{Ind}(\mathcal{D}) \sqcup \{0\} \to \mathbb{V}$ to satisfy the continuous deformed mesh relations (Definition 3.18).

Definition 3.17. Let $\underline{c} : \text{Ind}(\mathcal{D}) \sqcup \{0\} \to \mathbb{R}$ be a function such that $\underline{c}(0) = 0$. We say $\underline{c}$ is *permissible* if for every tilting rectangle $XYWZ$ in $\mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, the surface integral

$$\int_{XYWZ} \underline{c}$$

over $XYWZ$ yields a real number.

We now define what it means for a function Φ to satisfy the continuous deformed mesh relations. The reader is invited to compare the following definition with constructions preceding Theorem 2.3. When we say a real vector space $\mathbb{V}$ has coordinates indexed by a set Ω, we mean $\mathbb{V} = \prod_{\Omega} \mathbb{R}$.

Definition 3.18. Let $\mathcal{Z}$ be a zigzag in $\mathcal{D}$, $\underline{c} : \text{Ind}(\mathcal{D}) \sqcup \{0\} \to \mathbb{R}$ a permissible function, and $\mathbb{V}$ a real vector space of arbitrary dimension whose coordinates are indexed by a set Ω. Suppose $\Phi : \text{Ind}(\mathcal{D}) \sqcup \{0\} \to \mathbb{V}$ is a function such that for every tilting rectangle $XYWZ$ in $\mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\omega \in \Omega$ the following equation is satisfied:

$$\Phi(X)(\omega) + \Phi(W)(\omega) = \Phi(Y)(\omega) + \Phi(Z)(\omega) + \int_{XYWZ} \underline{c}.$$

Then we say Φ *satisfies the continuous deformed mesh relations over $\underline{c}$.*

In Section 6 we work with $\mathbb{V} = \mathbb{R}$. However, to adopt a framework that does not require modification in other contexts, we also allow $\mathbb{V}$ to be an arbitrary product of copies of $\mathbb{R}$.

Remark 3.19. If $\underline{c}$ in Definition 3.18 is the constant function at 0, then Φ satisfies the continuous mesh relations as in Section 2.3.

Proposition 3.20. *Let $\mathcal{Z}$ be a zigzag in $\mathcal{D}$ and W an indecomposable in a patch between $\mathcal{Z}$ and $\mathfrak{Q}(\mathcal{Z})$. If W is not in $\mathcal{Z}$, there exist X, Y, and Z in $Ob(\mathcal{Z}) \sqcup \{0\}$ such that $XYWZ$ is a tilting rectangle. If W is instead an indecomposable in a patch between $\mathfrak{Q}^{-1}(\mathcal{Z})$ and $\mathcal{Z}$, but not in $\mathcal{Z}$, then there exist X, Y, and Z in $Ob(\mathcal{Z}) \sqcup \{0\}$ such that $WYXZ$ is a tilting rectangle.*

Proof. First assume W is contained in a patch between $\mathcal{Z}$ and $\mathfrak{Q}(\mathcal{Z})$. If W is contained in a rectangular patch then the indecomposables Y and Z are obtained by intersecting $\mathcal{Z}$ with the lines of slope ± 1 that intersect at W. The indecomposable X is the left vertex shared by the line segments

used to create the rectangle.
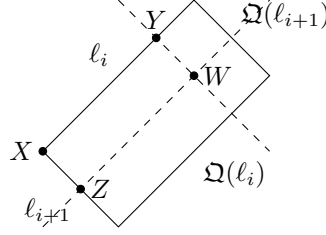
Note that if W is the intersection of $\mathfrak{Q}(\ell_{i+1})$ and $\mathfrak{Q}(\ell_i)$ then Y is the top point of the rectangle and Z is the bottom point. In this case Y or Z may be 0.

If W is contained in a triangular patch then Y and Z are obtained by intersecting $\mathcal{Z}$ and the base of the triangle with the lines of slope ± 1 that intersect at W. At least one of these will be 0.

Let h be the line containing W that intersects the base of the triangle to the left of W. Then X is obtained by intersecting $\mathcal{Z}$ with the line perpendicular to h that intersects at the same point on the base of the triangle.

or
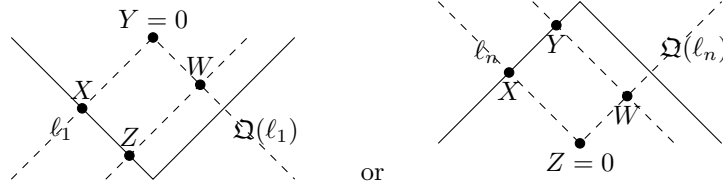
If W is in $\mathfrak{Q}(\ell_i)$ and $\mathcal{Z}$ has one line segment then both Y and Z are 0.

If instead W is contained in a patch between $\mathfrak{Q}^{-1}(\mathcal{Z})$ and $\mathcal{Z}$, then the argument is the same using the symmetric geometry. $\qquad\square$

Definition 3.21. Let $\mathcal{Z}$ be a zigzag in $\mathcal{D}$ and let W be an indecomposable in $\mathcal{D}$. We say W *can be quilted from* $\mathcal{Z}$ if there exists a nonnegative $n \in \mathbb{Z}$ such that W is in a patch of $\mathfrak{Q}^n(\mathcal{Z})$. We say W *can be inverse quilted from* $\mathcal{Z}$ if there exists a negative $n \in \mathbb{Z}$ such that W is in a patch of $\mathfrak{Q}^n(\mathcal{Z})$.

The following lemma shows that for any zigzag $\mathcal{Z}$ in $\mathcal{D}$ and any indecomposable W in $\mathcal{D}$, W can be quilted or inverse quilted from $\mathcal{Z}$.

Lemma 3.22. *Let $\mathcal{Z}$ be a zigzag in $\mathcal{D}$. For every indecomposable W in $\mathcal{D}$, there exists $n \in \mathbb{Z}$ such that W is in a patch of $\mathfrak{Q}^n(\mathcal{Z})$.*

Proof. Let y_W be the y-coordinate of W. Then there exist X and X' in $\mathcal{Z}$ where the y-value of X is y_W and the y-value of X' is $-y_W$. If $y_W = 0$ then $X = X'$.

Let x_W, x_0, and x'_0 be the x-values of respectively W, X, and X'. Since $|x_0 - x'_0| < \pi$, we have $x'_0 < x_0 + \pi$ and $x_0 < x'_0 + \pi$. So, for every $m \in \mathbb{Z}$ define

$$x_m = \begin{cases} m\pi + x_0 & m \text{ is even} \\ m\pi + x'_0 & m \text{ is odd}. \end{cases}$$

Then there exists $m \in \mathbb{Z}$ such that $x_m \leq x_W < x_{m+1}$.

Let j be the number of line segments in $\mathcal{Z}$. Then $\mathfrak{Q}^j(\mathcal{Z}[m]) = \mathcal{Z}[m+1]$ by Theorem 3.16. For each $0 \leq i < j$ let X_i be the object on $\mathfrak{Q}^i(\mathcal{Z}[m])$ with y-coordinate y_W. Let $x_{m,i}$ be the x-coordinate of X_i and set $x_{m,j} = x_{m+1}$. Then there exists $0 \leq i_W < j$ such that $x_{m,i_W} \leq x_W \leq x_{m,i_W+1}$.

It follows that W is in a patch of $\mathfrak{Q}^{i_W}(\mathcal{Z}[m])$, so W is in a patch of $\mathfrak{Q}^n(\mathcal{Z})$, where $n = jm + i_W$. $\quad\square$

Let $\underline{c} : \mathrm{Ind}(\mathcal{D}) \sqcup \{0\} \to \mathbb{R}$ be a permissible function, $\mathbb{V}$ a real vector space whose coordinates are indexed by a set Ω, $\mathcal{Z}$ a zigzag in $\mathcal{D}$, and $\Phi : Ob(\mathcal{Z}) \to \mathbb{V}$ a function. For each indecomposable $W \notin \mathcal{Z}$ in a patch between $\mathcal{Z}$ and $\mathfrak{Q}(\mathcal{Z})$ (respectively between $\mathfrak{Q}^{-1}(\mathcal{Z})$ and $\mathcal{Z}$), there is a unique tilting rectangle $\Diamond = XYWZ$ (respectively $\Diamond = WYXZ$) as in Proposition 3.20.

Definition 3.23. With the same notation and setting as above, the *quilting value of W over $\underline{c}$* is the vector $\Phi(W)$ in $\mathbb{V}$, where the value for each coordinate $\omega \in \Omega$ is given by

$$\Phi(W)(\omega) := \Phi(Y)(\omega) + \Phi(Z)(\omega) - \Phi(X)(\omega) + \int_{\Diamond} \underline{c}.$$

Proposition 3.24. *Let $\underline{c} : \mathrm{Ind}(\mathcal{D}) \sqcup \{0\} \to \mathbb{R}$ be a permissible function and $\mathbb{V}$ a real vector space with coordinates indexed by a set Ω. For a zigzag $\mathcal{Z}$ and function $\Phi : \mathrm{Ind}(\mathcal{Z}) \to \mathbb{V}$, there exists a unique extension of Φ to $\mathrm{Ind}(\mathcal{D}) \sqcup \{0\}$ that satisfies the continuous deformed mesh relations over $\underline{c}$.*

Proof. Using Definition 3.23 we may extend Φ to all the patches of $\mathcal{Z}$ and, on these patches, the extension satisfies the continuous deformed mesh relations over $\underline{c}$. Similarly, we may extend Φ to all the patches of $\mathfrak{Q}^{-1}(\mathcal{Z})$. Notice these extensions are unique.

By Lemma 3.22, each indecomposable W in $\mathcal{D}$ is in a patch of $\mathfrak{Q}^n(\mathcal{Z})$, for some $n \in \mathbb{Z}$. We recursively use this argument to obtain the desired extension of Φ. In particular, this uniquely defines $\Phi(W)$. Therefore, we may extend Φ to $\mathrm{Ind}(\mathcal{D})$ as stated in the proposition. $\square$

4. Connections to representation theory

Our goal in this section is to highlight some fundamental connections between our construction and the representation theory of quivers. More specifically, in Sections 4.2 and 4.3 we respectively introduce the notions of g-vectors and dimension vectors in our setting. Furthermore, from the results of Sections 4.1 and 4.4, we observe that for any zigzag $\mathcal{Z}$, there is a t-structure in $\mathcal{D}$ whose heart is analogous to the category of finitely generated representations of an A_n quiver. In particular, by the end of this section the reader observes that the subspace of $\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ corresponding to the indecomposables in $\mathcal{D}^{\heartsuit}$ shares many properties with the Auslander–Reiten quiver of a type A_n quiver (for further details, see Section 4.4).

4.1. **t-structures.** For a zigzag $\mathcal{Z}$ in $\mathcal{D}$, define the full subcategories

$$\mathcal{D}^{\leq 0} = \mathrm{add}\{X \mid X \text{ can be quilted from } \mathcal{Z}\}$$
$$\mathcal{D}^{\geq 0} = \mathrm{add}\left(\{Y \mid Y \text{ can be inverse quilted from } \mathcal{Z}[1]\} \setminus \mathcal{Z}[1]\right).$$

We have a t-structure because the following hold.

- $\mathcal{D}^{\leq 0}$ is closed under $[1]$ and $\mathcal{D}^{\geq 0}$ is closed under $[-1]$.
- For any indecomposables X in $\mathcal{D}^{\leq 0}$ and Y in $\mathcal{D}^{\geq 0}$, we have $\mathrm{Hom}_{\mathcal{D}}(X, Y[-1]) = 0$. Since $\mathcal{D}$ is Krull–Schmidt, this extends to all X in $\mathcal{D}^{\leq 0}$ and Y in $\mathcal{D}^{\geq 0}$.
- Every indecomposable in $\mathcal{D}$ belongs to at least one of $\mathcal{D}^{\leq 0}$ or $\mathcal{D}^{\geq 0}$. This immediately yields that every object E in $\mathcal{D}$ belongs to a distinguished triangle

$$X \longrightarrow E \longrightarrow Y[-1] \longrightarrow X[1],$$

  where X is in $\mathcal{D}^{\leq 0}$ and Y is in $\mathcal{D}^{\geq 0}$.

Notice that the heart $\mathcal{D}^{\heartsuit} := \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ does not contain $\mathcal{Z}[1]$, but it does contain all other indecomposables that can be both quilted from $\mathcal{Z}$ (including $\mathcal{Z}$) and inverse quilted from $\mathcal{Z}[1]$. We observe that $\mathcal{D}^{\heartsuit}$ is similar to the categories of representations of continuous quivers in [IT15, IRT19, R19], where the projective representations are those on $\mathcal{Z}$ (Proposition 4.4). See Section 4.4 for further discussion. Representations of continuous quivers connect our interpretation to the construction given in [BD+18].

We remark that once we have chosen $\mathcal{Z}$, we have no further choices regarding our t-structure. This implies that the heart is in some sense canonical.
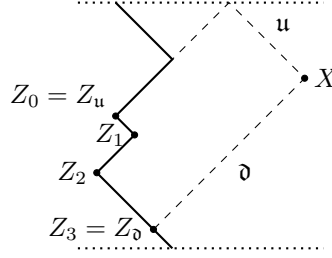
FIGURE 7. An example $\mathcal{Z}$ and X with $\mathfrak{u}$, $\mathfrak{d}$, $Z_{\mathfrak{u}}$, and $Z_{\mathfrak{d}}$. Each of the Z_i are also shown. The region bounded by $\mathfrak{u}$, $\mathfrak{d}$, and the zigzag will be of use in Section 6.1.

4.2. $g_{\mathcal{Z}}$-**vectors.** Let $\mathcal{Z}$ be a zigzag in $\mathcal{D}$. We now show that the heart $\mathcal{D}^{\heartsuit}$ of the t-structure obtained from $\mathcal{Z}$ in Section 4.1 has $g_{\mathcal{Z}}$-vectors, which behave like g-vectors in the classical sense.

Before we state the next definition, recall that each object Z in $\mathcal{Z}$ is an indecomposable in $\mathcal{D}$.

Definition 4.1 ($g_{\mathcal{Z}}$-vectors). Let $\mathbb{V}$ be the real vector space whose coordinates are indexed by $Ob(\mathcal{Z})$. For each object Z in $\mathcal{Z}$, set $g_{\mathcal{Z}}(Z)$ to be the vector in $\mathbb{V}$ whose coordinates are 1 in the Z-coordinate and 0 elsewhere. Let $\underline{c}$ be the 0 function. We uniquely extend $g_{\mathcal{Z}}$ to all of $\mathcal{D}$ as in Proposition 3.24. For an indecomposable X in $\mathcal{D}^{\heartsuit} \sqcup \mathcal{Z}[1]$, the $g_{\mathcal{Z}}$-*vector of* X is defined to be $g_{\mathcal{Z}}(X)$.

We now give an explicit description of the $g_{\mathcal{Z}}$-vectors in $\mathcal{D}^{\heartsuit} \sqcup \mathcal{Z}[1]$; namely, in Proposition 4.2, we show that each such $g_{\mathcal{Z}}$-vector $g_{\mathcal{Z}}(X)$ is a finite sum of $g_{\mathcal{Z}}$-vectors of objects in $\mathcal{Z}$.

For each indecomposable in $\mathcal{Z}$, the associated $g_{\mathcal{Z}}$-vector is already defined. Therefore, to explicitly describe the $g_{\mathcal{Z}}$-vector of every indecomposable X in $\mathcal{D}^{\heartsuit}$, we only need to treat those indecomposable X in $\mathcal{D}^{\heartsuit}$ such that X is not in $\mathcal{Z}$. For every such indecomposable X consider the rays $\mathfrak{u}$ and $\mathfrak{d}$ in $\mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, respectively with slopes -1 and $+1$, which emanate from X and propagate in the negative x-direction. (For a graphical depiction, see Figure 7.) Each of these rays $\mathfrak{u}$ and $\mathfrak{d}$ may "bounce" off one of the horizontal lines $y = \frac{\pi}{2}$ or $y = -\frac{\pi}{2}$ at most once before they intersect $\mathcal{Z}$. Further, we associate a unique point $Z_{\mathfrak{u}}$ in $\mathcal{Z}$ to the ray $\mathfrak{u}$ as follows. (The point $Z_{\mathfrak{d}}$ can be described analogously.)

(1) Suppose $\mathfrak{u}$ bounces off $y = \frac{\pi}{2}$ *and* intersects $\mathcal{Z}$ at a right vertex Z (Definition 3.1). Then we define $Z_{\mathfrak{u}}$ to be the adjacent left vertex below Z or the intersection between $\mathfrak{u}$ and $\mathfrak{d}$ to the left of X, whichever is closer to X.

(2) If $\mathfrak{u}$ does not bounce off $y = \frac{\pi}{2}$ *or* does not intersect $\mathcal{Z}$ at a right vertex of $\mathcal{Z}$, we define $Z_{\mathfrak{u}}$ to be the rightmost intersection between $\mathcal{Z}$ and $\mathfrak{u}$.

By Euclidean geometry and the triangulated structure of $\mathcal{D}$ (Section 2.2.2), $Z_{\mathfrak{u}} = Z_{\mathfrak{d}}$ if and only if $X \in \mathcal{Z}[1]$.

Suppose $X \notin \mathcal{Z}[1]$. Consider the (possibly empty) set of all left and right vertices of $\mathcal{Z}$ whose y-coordinates are strictly between those of $Z_{\mathfrak{u}}$ and $Z_{\mathfrak{d}}$. Enumerate these left and right vertices as Z_i, starting with Z_1, where $i < i'$ if the y-coordinate of Z_i is greater than the y-coordinate of $Z_{i'}$. Let j be the number of such Z_i's, $Z_0 = Z_{\mathfrak{u}}$, and $Z_{j+1} = Z_{\mathfrak{d}}$. For $i \notin [0, j+1]$, set $Z_i = 0$.

Proposition 4.2. *Let X be an indecomposable in $\mathcal{D}^{\heartsuit} \sqcup \mathcal{Z}[1]$, but not in $\mathcal{Z}$. Then the $g_{\mathcal{Z}}$-vector of X is given by*
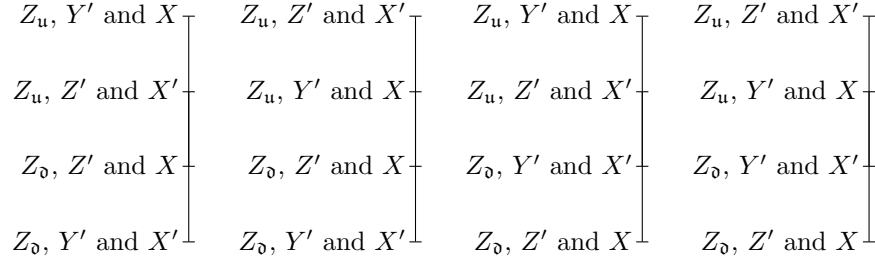
$$g_{\mathcal{Z}}(X) = \begin{cases} \displaystyle\sum_{\text{odd } i} g_{\mathcal{Z}}(Z_i) - \sum_{\text{even } i} g_{\mathcal{Z}}(Z_i) & Z_0 \text{ is left of } Z_1 \\ \displaystyle\sum_{\text{even } i} g_{\mathcal{Z}}(Z_i) - \sum_{\text{odd } i} g_{\mathcal{Z}}(Z_i) & Z_0 \text{ is right of } Z_1 \\ -g_{\mathcal{Z}}(X[-1]) & X \in \mathcal{Z}[1]. \end{cases}$$

Proof. First, we treat the case $X \in \mathcal{Z}[1]$. In this case, note that $Z_{\mathfrak{u}} = Z_{\mathfrak{d}} = X[-1]$. Consider the tilting rectangle whose left corner is $X[-1]$ and right corner is X, where the top and bottom corners lie on $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$, respectively. Thus, $g_{\mathcal{Z}}(X[-1]) + g_{\mathcal{Z}}(X) = 0$ implies the desired result.

Now suppose $X \notin \mathcal{Z}[1]$. Let n_ℓ be the number of line segments of $\mathcal{Z}$. We proceed by strong induction on $n < n_\ell$, starting with $n = 0$. If X is in a patch of $\mathcal{Z} = \mathfrak{Q}^0(\mathcal{Z})$, the result follows from Definition 3.23.

Suppose the result holds for all X in any patch of $\mathfrak{Q}^m_{\mathcal{Z}}$, for all $0 \leq m \leq n < n_\ell - 1$. Let X be in a patch of $\mathfrak{Q}^{n+1}(\mathcal{Z})$, and define $Z_{\mathfrak{u}}$, $Z_{\mathfrak{d}}$, and the Z_i as above. Further, we can assume X is not in $\mathfrak{Q}^{n+1}(\mathcal{Z})$, as otherwise X is in a patch of $\mathfrak{Q}^n(\mathcal{Z})$ and we are done.

Since $X \notin \mathcal{Z}[1]$, we know $Z_{\mathfrak{u}} \neq Z_{\mathfrak{d}}$ and the y-coordinate of $Z_{\mathfrak{u}}$ is greater than the y-coordinate of $Z_{\mathfrak{d}}$. Furthermore, since X is in a patch of $\mathfrak{Q}^{n+1}(\mathcal{Z})$, there exist X', Y', and Z' in $\mathfrak{Q}^{n+1}(\mathcal{Z}) \sqcup \{0\}$ such that $X'Y'XZ'$ is a tilting rectangle. By induction we know the proposition holds for X', Y', and Z'. Note that the rays $\mathfrak{d}$ coming from Y' and X' will determine the same $Z_{\mathfrak{d}}$ for both objects. Similarly, $Z_{\mathfrak{u}}$ is the same for Z' and X'. Moreover, $Z_{\mathfrak{u}}$ is the same for X and Y'; $Z_{\mathfrak{d}}$ is the same for X and Z'. The y-coordinates of these four objects in $\mathcal{Z}$ will all be distinct since X is not in $\mathfrak{Q}^{n+1}(\mathcal{Z})$. Schematically, there are four cases of the y-coordinates based on which rays (if any) bounce off the boundary:

$$
\begin{array}{llll}
Z_{\mathfrak{u}}, Y' \text{ and } X & Z_{\mathfrak{u}}, Z' \text{ and } X' & Z_{\mathfrak{u}}, Y' \text{ and } X & Z_{\mathfrak{u}}, Z' \text{ and } X' \\
Z_{\mathfrak{u}}, Z' \text{ and } X' & Z_{\mathfrak{u}}, Y' \text{ and } X & Z_{\mathfrak{u}}, Z' \text{ and } X' & Z_{\mathfrak{u}}, Y' \text{ and } X \\
Z_{\mathfrak{d}}, Z' \text{ and } X & Z_{\mathfrak{d}}, Z' \text{ and } X & Z_{\mathfrak{d}}, Y' \text{ and } X' & Z_{\mathfrak{d}}, Y' \text{ and } X' \\
Z_{\mathfrak{d}}, Y' \text{ and } X' & Z_{\mathfrak{d}}, Y' \text{ and } X' & Z_{\mathfrak{d}}, Z' \text{ and } X & Z_{\mathfrak{d}}, Z' \text{ and } X
\end{array}
$$

Therefore, since
$$
g_{\mathcal{Z}}(X) = g_{\mathcal{Z}}(Y') + g_{\mathcal{Z}}(Z') - g_{\mathcal{Z}}(X'),
$$
we see the proposition follows. $\qquad\square$

4.3. **Dimension vectors.** Again let $\mathcal{Z}$ be a zigzag and $\mathcal{D}^{\heartsuit}$ be the heart of the t-structure obtained from $\mathcal{Z}$, as in Section 4.1. Now we show that for each indecomposable X in $\mathcal{D}^{\heartsuit}$, there is a notion of dimension vector analogous to the discrete case.

First we introduce a partial order on $\mathcal{Z}$: If Z and Z' belong to the same line segment of $\mathcal{Z}$, we put $Z \leq Z'$ provided that the x-coordinate of Z is no larger than the x-coordinate of Z'. If Z and Z' are not on the same line segment in $\mathcal{Z}$, then Z and Z' are not comparable. Then, for each Z in $\mathcal{Z}$, define $\underline{\dim}_{\mathcal{Z}}(Z)$ in $\prod_{Ob(\mathcal{Z})} \mathbb{R}$ to be 1 on each Z'-coordinate with $Z' \leq Z$, and 0 elsewhere. Again by Proposition 3.24, we extend $\underline{\dim}_{\mathcal{Z}}$ uniquely to a function $\underline{\dim}_{\mathcal{Z}} : \mathcal{D} \to \prod_{Ob(\mathcal{Z})} \mathbb{R}$. Now, for each indecomposable object X in $\mathcal{D}^{\heartsuit}$, define the *dimension vector of X with respect to $\mathcal{Z}$* to be $\underline{\dim}_{\mathcal{Z}}(X)$.

Proposition 4.3. *Let X be an indecomposable object in $\mathcal{D}^{\heartsuit}$. Then each coordinate of $\underline{\dim}_{\mathcal{Z}}(X)$ is 0 or 1. Moreover, $\{Z \in \mathrm{Ind}(\mathcal{Z}) \mid Z\text{-coordinate of } \underline{\dim}_{\mathcal{Z}}(X) \text{ is } 1\}$ forms a connected set, where $\mathrm{Ind}(\mathcal{Z}) \subsetneq \mathbb{R}^2$ has the subspace topology.*

Proof. If $X \in \mathcal{Z}$, the proposition follows by the definition of $\underline{\dim}_{\mathcal{Z}}$. We use the same notation as in Section 4.2. In particular, let $Z_{\mathfrak{u}}$, $Z_{\mathfrak{d}}$, and the Z_i be as before. For the case where $X \notin \mathcal{Z}$, we know that Z_0 and Z_1 are distinct and we can assume that Z_0 is to the left of Z_1 (the other case is similar). We further remark that the proof of Proposition 4.2 does not rely on the values of $g_{\mathcal{Z}}$. Thus,
$$
\underline{\dim}_{\mathcal{Z}}(X) = \sum_{\text{odd } i} \underline{\dim}_{\mathcal{Z}}(Z_i) - \sum_{\text{even } i} \underline{\dim}_{\mathcal{Z}}(Z_i) = \sum_{i \in \mathbb{Z}} (-1)^{i+1} \underline{\dim}_{\mathcal{Z}}(Z_i).
$$

We now prove the result by induction on the number of Z_i's. We start with $Z_0 = Z_{\mathfrak{u}}$ and $Z_1 = Z_{\mathfrak{d}}$. In this case, the result is straightforward to check.

For the induction step on $j + 1 \geq 2$, let $\mathcal{S}$ be the set of $Z \in \mathcal{Z}$ such that the Z-coordinate of $\sum_{i=1}^j (-1) \underline{\dim}_{\mathcal{Z}}(Z_i)$ is 1. Assume $\mathcal{S}$ is connected and that the Z-coordinate of $\sum_{i=1}^j (-1) \underline{\dim}_{\mathcal{Z}}(Z_i)$

is 0 for all $Z \notin \mathcal{S}$. There are even and odd cases for the induction step; we first consider the even case. If $j + 1$ is even, then Z_j is a right vertex and $\mathcal{S} \setminus \{Z \mid Z \leq Z_{j+1}\}$ is also a connected set. If $j + 1$ is odd, then $Z_j \notin \mathcal{S}$ is a left vertex and $\mathcal{S} \cup \{Z \mid Z \leq Z_{j+1}\}$ is a connected set. In both cases, $\sum_{i=1}^{j}(-1)^{i+1}\underline{\dim}_{\mathcal{Z}}(Z_i)$ is 1 precisely on $\mathcal{S}$ and 0 elsewhere. $\qquad\square$

4.4. **Continuous representations and $\mathcal{D}^{\heartsuit}$.** In this brief section we discuss how $\mathcal{D}^{\heartsuit}$ is similar to $\mathrm{rep}(A_n)$ for a type A_n quiver, and thus may be thought of as a category of continuous representations. We have already shown that the $g_{\mathcal{Z}}$-vectors (Section 4.2) and dimension vectors (Section 4.3) in $\mathcal{D}^{\heartsuit}$ behave similarly to g-vectors and dimension vectors, respectively, in $\mathrm{rep}(A_n)$.

For a type A_n quiver (Section 2.1), the projective indecomposables in $\mathrm{rep}(A_n)$ form a zigzag shape in the (augmented) Auslander–Reiten quiver. The following proposition shows that the indecomposables in the zigzag $\mathcal{Z}$ are exactly the projective indecomposable objects in $\mathcal{D}^{\heartsuit}$.

Proposition 4.4. *An indecomposable Z is projective in $\mathcal{D}^{\heartsuit}$ if and only if $Z \in \mathrm{Ind}(\mathcal{Z})$.*

Proof. Let Z be an indecomposable in $\mathcal{Z}$ and $X \to Y \to Z$ be a short exact sequence in $\mathcal{D}^{\heartsuit}$. Note that if an indecomposable W in $\mathcal{D}^{\heartsuit}$ is not on $\mathcal{Z}$ then $\mathrm{Hom}_{\mathcal{D}^{\heartsuit}}(W, Z) = 0$. This implies X and Y consist of indecomposable summands on $\mathcal{Z}$ as well. Further note that for any pair of indecomposables W, W' in $\mathcal{Z}$, if W and W' are not on the same line segment then $\mathrm{Hom}_{\mathcal{D}^{\heartsuit}}(W, W') = 0$ (see Hom supports in Figure 3). By the triangulated structure proven in [IT15], and thus the abelian structure in $\mathcal{D}^{\heartsuit}$, any such short exact sequence must be split. Thus, Z is projective in $\mathcal{D}^{\heartsuit}$.

Let X be an indecomposable in $\mathcal{D}^{\heartsuit}$, but not in $\mathcal{Z}$. Then we may find a small enough nondegenerate tilting rectangle $\Diamond$ in $\mathcal{D}^{\heartsuit}$ to the right of $\mathcal{Z}$, such that the right corner of $\Diamond$ is X. This distinguished triangle, and thus the short exact sequence in $\mathcal{D}^{\heartsuit}$, does not split. $\qquad\square$

In the following theorem, we capture some of the main properties of $\mathcal{D}^{\heartsuit}$. Since the results follow from Propositions 4.2 and 4.4, we omit the proof.

Theorem 4.5. *The abelian category $\mathcal{D}^{\heartsuit}$ has enough projectives, it is Krull–Schmidt, and every indecomposable object is finitely generated. From the construction of $\mathcal{D}$, it follows that $\mathrm{Ext}^{i}_{\mathcal{D}^{\heartsuit}}(X, Y) = 0$ for $i > 1$, thus $\mathcal{D}^{\heartsuit}$ is of global dimension 1. Furthermore, the isomorphism classes of indecomposable objects of $\mathcal{D}$ are given by shifts of those in $\mathcal{D}^{\heartsuit}$.*

From the preceding theorem, observe that one can think of $\mathcal{D}^{\heartsuit}$ as the category of certain finitely generated representations of a continuous quiver whose orientation is inherited by the partial order on $\mathcal{Z}$. The reader is referred to [IRT19] for a detailed introduction to continuous quivers of type A. Our results are inspired by, but not reliant upon, that work.

5. **T**-CLUSTERS

In this section, we present a continuous generalization of the clusters and compatibility in [BD+18, Section 2]. This will be used in our construction of a continuous analogue of the ABHY associahedron for type A quivers, as we discuss in Section 6.

For the remainder of the paper, let $\mathcal{Z}$ be a fixed zigzag in $\mathcal{D}$ (Definition 3.1). Recall that $\mathcal{D}^{\heartsuit}$ denotes the heart of the t-structure in $\mathcal{D}$, as described in Section 4.1. We consider the following subcategory of $\mathcal{D}$.

Notation 5.1. Let $\mathcal{C}_{\mathcal{Z}} := \mathrm{add}(\mathrm{Ind}(\mathcal{D}^{\heartsuit}) \sqcup \mathrm{Ind}(\mathcal{Z}[1]))$. Namely, $\mathcal{C}_{\mathcal{Z}}$ is the full subcategory of $\mathcal{D}$ whose objects are finite direct sums of indecomposable objects in $\mathcal{D}^{\heartsuit}$ and $\mathcal{Z}[1]$.

5.1. **Compatibility.** To generalize the compatibility in [BD+18] to the continuous version, we now define compatibility in $\mathcal{C}_{\mathcal{Z}}$, making use of the continuous deformed mesh relations (Definition 3.18).

Definition 5.2. Let X and Y be indecomposable objects in $\mathcal{C}_{\mathcal{Z}}$. We say X and Y are *incompatible* if there exists a distinguished triangle in $\mathcal{D}$ of one of the following forms:

$$X \longrightarrow E \longrightarrow Y \longrightarrow X[1] \qquad \text{or} \qquad Y \longrightarrow E \longrightarrow X \longrightarrow Y[1].$$

Otherwise, we say X and Y are *compatible*.

Remark 5.3. Note that Definition 5.2 implies that each indecomposable $X \in \mathcal{C}_\mathcal{Z}$ is compatible with itself. Recall that in Section 2.1, for two vertices of $\widetilde{\Gamma}_\mathcal{C}$, we considered the notion of compatibility based on the associated cluster variables in the cluster algebra of type A_n. From the tilting rectangles (Definition 2.5), one observes that the compatibility condition in Definition 5.2 is analogous to that in Section 2.1, described in Remark 2.2. In particular, two indecomposable objects X and Y in $\mathcal{C}_\mathcal{Z}$ are incompatible if and only if there exists a tilting rectangle in $\mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ whose left and right corners are X and Y (or Y and X).

We also note that if X is an indecomposable object in $\mathcal{Z}$, then X and $X[1]$ are incompatible. This is because there is a distinguished triangle $X \to 0 \to X[1] \to X[1]$ in $\mathcal{D}$, which corresponds to a tilting rectangle in $\mathbb{R} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. By Proposition 4.4, this recovers a well known property of cluster structures in the categorical settings: Every indecomposable projective P is incompatible with its shift $P[1]$.

Finally, we remark that our compatibility condition differs from the condition used by Igusa and Todorov in [IT15], which is based on Ext spaces. It also differs from the condition for **E**-clusters used by Igusa, Todorov, and the fourth author in [IRT20], which is based on the Euler product.

5.2. **T-clusters.** In this subsection, we define **T**-clusters. Since our compatibility condition is based on tilting rectangles, we use the prefix **T**-, which also distinguishes our compatibility condition from those in [IT15] and [IRT20]. While the **T**-clusters do not form a cluster structure in the sense of [BI+09], they have many properties of clusters. In particular, in Section 5.3 we show that mutation of **T**-clusters is relatively well behaved.

Definition 5.4. A **T**-*cluster* $\mathcal{T}$ is a maximal collection of pairwise compatible indecomposable objects in $\mathcal{C}_\mathcal{Z}$. That is, a collection of indecomposable objects $\mathcal{T}$ in $\mathcal{C}_\mathcal{Z}$ is a **T**-cluster if every pair X and Y in $\mathcal{T}$ are compatible, and if for each $Z \notin \mathcal{T}$ there exists $X \in \mathcal{T}$ such that X and Z are incompatible.

Example 5.5. We provide a list of basic examples of **T**-clusters.

(1) The set of indecomposable objects in $\mathcal{Z} \subsetneq \mathcal{C}_\mathcal{Z}$ is a **T**-cluster. Similarly, the set of indecomposable objects in $\mathcal{Z}[1] \subsetneq \mathcal{C}_\mathcal{Z}$ is a **T**-cluster.

(2) Let ℓ be a smooth curve in $\mathcal{D}$ such that the slope of ℓ at each point is less than -1, greater than $+1$, or equal to ∞ and, for all $a \in (-\frac{\pi}{2}, \frac{\pi}{2})$, there is an $A \in \ell$ such that the y-coordinate of A is a. If $\ell \subsetneq \mathrm{Ind}(\mathcal{C}_\mathcal{Z})$, then $\mathcal{T} = \{A \mid A \in \ell\}$ is a **T**-cluster.

(3) Let $\mathcal{Z}'$ be a zigzag in $\mathcal{D}$ distinct from $\mathcal{Z}$ and $\mathcal{Z}[1]$. If $\mathrm{Ind}(\mathcal{Z}') \subsetneq \mathrm{Ind}(\mathcal{C}_\mathcal{Z})$, then $\mathcal{Z}'$ is a **T**-cluster.

Remark 5.6. Note that all of the **T**-clusters in Example 5.5 are connected as subsets of $\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$. We warn the reader that this is not always the case. In fact, **T**-clusters may be *totally disconnected* in $\mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$. We further discuss this phenomenon through an example in Section 5.2.1.

5.2.1. *A totally disconnected example.* Suppose $\mathcal{Z}$ consists of one line segment of slope 1. Without loss of generality, assume the indecomposable corresponding to the point $(0,0)$ is in $\mathcal{Z}$. We use a recursive process to modify $\mathcal{Z}$ and obtain a totally disconnected **T**-cluster. In fact, we produce a sequence of **T**-clusters $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ where the limit of the process, denoted by $\mathcal{T}_\infty$, is a totally disconnected set which is maximally compatible.

As already stated in Example 5.5, $\mathcal{Z}$ is a **T**-cluster. Set $\mathcal{T}_0 = \mathcal{Z}$. Now we aim to explicitly describe $\mathcal{T}_1$, for which the reader may find it helpful to refer to Figure 8 as we construct this new **T**-cluster. Choose an integer-indexed subset $\mathcal{X} = \{(x_i, y_i) \mid i \in \mathbb{Z}\}$ of $\mathcal{T}_0 = \mathcal{Z}$ such that

- $y_i < y_{i+1}$, for all $i \in \mathbb{Z}$;
- $\mathcal{X}$ has two accumulation points, exactly at $(-\frac{\pi}{2}, -\frac{\pi}{2})$ and $(\frac{\pi}{2}, \frac{\pi}{2})$.

For each $i \in \mathbb{Z}$, let X_i be the indecomposable object in $\mathcal{Z}$ corresponding to (x_i, y_i), and note that each of these points has the form $X_i = (t_i\pi - \frac{\pi}{2}, t_i\pi - \frac{\pi}{2})$ for some $t_i \in (0, 1)$.
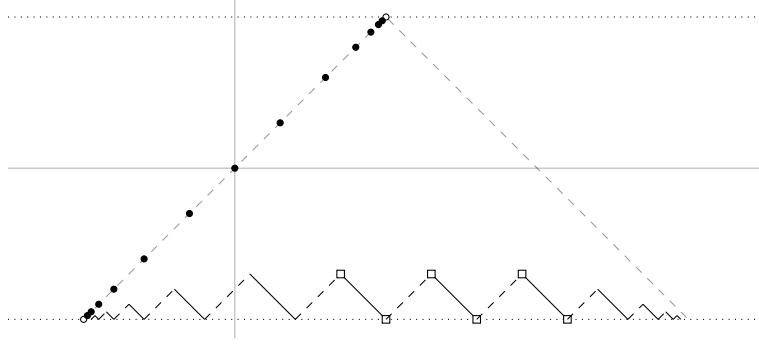
FIGURE 8. The indecomposable objects of $\mathcal{T}_1$ drawn as filled points in $\mathrm{Ind}(\mathcal{D}) \subsetneq \mathbb{R}^2$. The open circles are the described limit points, but are not objects in $\mathcal{D}$. Additionally, E_i and F_i are indicated by squares, for $i \in \{-1, 0, 1\}$.

For each $Y \in \mathcal{Z} \setminus \mathcal{X}$, there exist $i \in \mathbb{Z}$ and $s \in (0, 1)$ such that Y corresponds to the point

$$(1 - s)(x_i, y_i) + s(x_{i+1}, y_{i+1}).$$

For each X_i, let

$$E_i := \left(x_{i+1} + (x_i + \frac{\pi}{2}), y_{i+1} - (y_i + \frac{\pi}{2}) \right),$$

$$F_i := \left(x_{i+1} + (x_i + \frac{\pi}{2}) + (x_{i+1} - x_i), -\frac{\pi}{2} \right).$$

Then, by $f_1(Y)$ we denote the indecomposable in $\mathcal{C}_{\mathcal{Z}}$ which corresponds to the point $(1-s)E_i + sF_i$. Now, we set

$$\mathcal{T}_1 := \{X_i\} \cup \{f_1(Y) \mid \text{for all } i \in \mathbb{Z}, Y \neq X_i\}.$$

As it can be seen via the dashed lines in Figure 8, from the construction it follows that each of the open line segments from E_i to F_i form the right side of an isosceles triangle. This triangle is similar to the one defined by $\mathcal{Z}$, $\mathcal{Z}[1]$, and part of $y = -\frac{\pi}{2}$. Inside each of the smaller triangles, we repeat the construction of $\mathcal{T}_1$, except scaled and reflected about the vertical axis of symmetry in the triangle. This produces $\mathcal{T}_2$, with its own set of smaller triangles.

We repeat the above process on the smaller and smaller triangles obtained. The limit of this process, which we denote by $\mathcal{T}_\infty$, contains all the discrete points from $\mathcal{T}_n$, for all $n \geq 1$. In the limit, the line segments between the E_i's and F_i's vanish and we are left with a totally disconnected (indeed discrete) set.

Finally, note that the construction guarantees that each $\mathcal{T}_n$ is a maximally compatible set in $\mathcal{C}_{\mathcal{Z}}$.

Proposition 5.7. *With the same notation as above, the set $\mathcal{T}_\infty$ is a **T**-cluster.*

Proof. Let W be an indecomposable in $\mathcal{C}_{\mathcal{Z}}$ such that W is not in $\mathcal{T}_\infty$. It is straightforward to check that if W is not in $\mathcal{T}_n$ for all $n \geq 0$, then there exists $X \in \mathcal{T}_\infty$ such that W and X are incompatible.

Suppose W is in $\mathcal{T}_n$ for some $n \geq 0$. Without loss of generality, suppose W is in $\mathcal{T}_n$ but not $\mathcal{T}_{n+1}$. Then W was on a line segment obtained from the construction of $\mathcal{T}_n$. Further, there is a sub-segment of a line segment in $\mathcal{T}_{n+1}$, whose complement is also a sub-segment, such that W is incompatible with each X in the sub-segment (see Figure 9). Up to symmetry, we have the following picture which contains only the relevant part of $\mathcal{T}_{n+1}$. An infinite sequence of points used to construct $\mathcal{T}_{n+2}$ approaches the upper end of the red line segment. Then there exist infinitely many such X_i in the sequence that are incompatible with W. Each of the X_i in the sequence is in $\mathcal{T}_{n+2}$ and in $\mathcal{T}_\infty$. This concludes the proof. $\qquad\square$
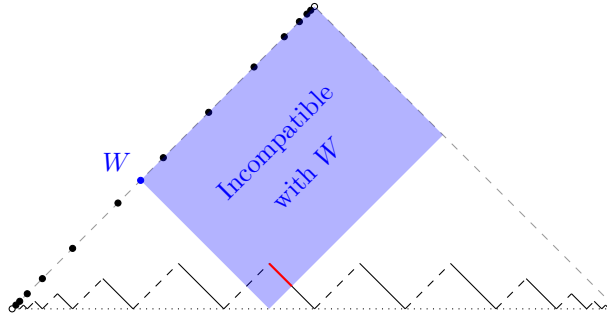
FIGURE 9. A visual aid to the proof of Proposition 5.7. There are infinitely many
objects in both the red line segment and $\mathcal{T}_\infty$.

5.3. **Mutation of T-clusters.** We now describe the notion of mutation for **T**-clusters, which we
call **T**-mutation (Definition 5.10). Again, the reason for the '**T**-' prefix is to indicate that our notion
of mutation is based on the compatibility condition, which itself relies on tilting rectangles. Before
we provide further details, we warn the reader that unlike in the standard setting of cluster algebras,
for an arbitrary object X in a **T**-cluster $\mathcal{T}$ there may not be a mutation at X.

To describe the notion of **T**-mutation, we need the following technical lemma.

Lemma 5.8. *Let $\mathcal{T}$ be a **T**-cluster. Suppose $X \in \mathcal{T}$ and $Y \notin \mathcal{T}$ such that $(\mathcal{T} \setminus \{X\}) \cup \{Y\}$ is a
compatible set. Then there exists a tilting rectangle in $\mathcal{C}_{\mathcal{Z}}$ whose left and right corners are X and Y
(or Y and X) and whose top and bottom corners are in $\mathcal{T}$. Moreover, for an indecomposable Z in
the interior or on a side of this tilting rectangle, if Z is not a corner vertex then it is not in $\mathcal{T}$.*

Proof. Because $\mathcal{T}$ is a **T**-cluster and further $(\mathcal{T} \setminus \{X\}) \cup \{Y\}$ is also a compatible set, it follows that
X and Y cannot be compatible. Hence, there exists a tilting rectangle in $\mathcal{C}_{\mathcal{Z}}$ whose left and right
corners are respectively X and Y, or dually Y and X. Since the cases are symmetric, without loss
of generality, we only treat the former case.

Let T_1 and T_2 respectively denote the top and bottom vertices of the rectangle. We aim to show
T_1 and T_2 belong to $\mathcal{T}$. Since $X \not\cong Y$, we must have $T_1 \not\cong T_2$. For the sake of contradiction, suppose
$T_1 \notin \mathcal{T}$ and further $T_1 \neq 0$. (The case for T_2 is similar.)

Since $T_1 \notin \mathcal{T}$, there must exist Z in $\mathcal{T}$ such that T_1 and Z are incompatible. Since X and T_1 are
joined by a diagonal, we conclude $Z \not\cong X$. Similarly, $Z \not\cong Y$. In the following diagram, consider the
regions labelled 1, 2, 3. (If $T_2 = 0$, we only have regions 1 and 3.)
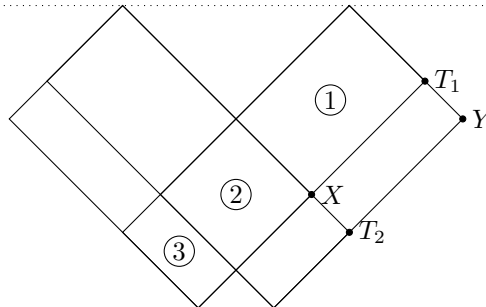
We show that Z cannot be to the left of T_1. This is because if Z is to the left of T_1, it must be
in regions 1, 2, or 3. But, Y is compatible with all objects in $\mathcal{T} \setminus \{X\}$, and therefore Z cannot be
in regions 1 or 2. Further, Z cannot be in regions 2 or 3, because Z is compatible with X.

Similarly, one can show that Z cannot be to the right of T_1. This gives the desired contradiction
and implies that T_1 belongs to $\mathcal{T}$.

To prove the last assertion of the lemma, suppose Z is on a side of the tilting rectangle but is not one of the corners. Then one may check Z is incompatible with either X or Y. If Z is in the interior of the tilting rectangle, it is compatible neither with X nor with Y. Thus, $Z \notin \mathcal{T}$. $\qquad \square$

Intuitively, if we may take out X and replace it with Y, then there is a tilting rectangle whose top and bottom corners are in $\mathcal{T}$ and nothing "between" the corners may be in $\mathcal{T}$. We use the preceding lemma and this intuition to prove the following proposition. In particular, the proposition implies that for a given **T**-cluster $\mathcal{T}$ and a given indecomposable object X in it, if the mutation at X is possible, then it is unique.

Proposition 5.9. *Let $\mathcal{T}$ be a **T**-cluster, and let X be in $\mathcal{T}$ such that there exists $Y \notin \mathcal{T}$ where $(\mathcal{T} \setminus \{X\}) \cup \{Y\}$ is a compatible set. Then the following are true.*

  (1) *The set $\mathcal{T}' := (\mathcal{T} \setminus \{X\}) \cup \{Y\}$ is a **T**-cluster.*
  (2) *If there exists Y' such that $(\mathcal{T} \setminus \{X\}) \cup \{Y'\}$ is also a compatible set, then $Y = Y'$.*

Proof. We start with the first statement. If the statement fails, there is an object Z in $\mathcal{C}_{\mathcal{Z}}$ such that Z is compatible with $\mathcal{T}'$ but not with $\mathcal{T}$. This implies that Z is incompatible with X but it is compatible with Y. By Lemma 5.8, there exists a tilting rectangle XT_1YT_2 (or YT_1XT_2) where T_1 and T_2 are in $\mathcal{T}$. By the same lemma, if M is a point in the tilting rectangle which is not on a corner, then M is incompatible either with X or with Y.

Without loss of generality (by symmetry), assume the tilting rectangle is XT_1YT_2. Consider the regions labelled 1 through 7 in the following picture.
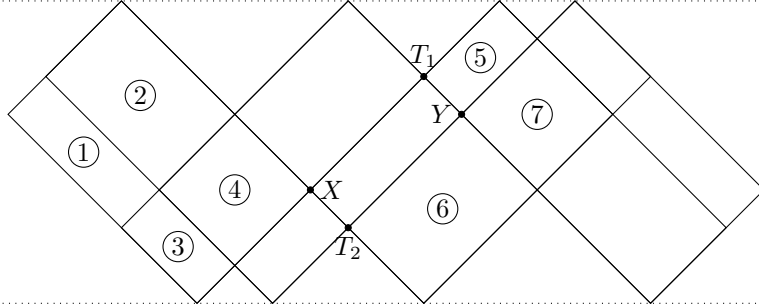
Note that Z must be in one of the labelled regions, or else Z is compatible with X or is incompatible with Y. First, observe that Z cannot be in region 1, because then Z would be to the left of $Y[-1]$ and thus not in $\mathcal{C}_{\mathcal{Z}}$. Next, Z cannot be in regions 2 or 3, because Z would be incompatible with T_2 or T_1, respectively. Further, Z cannot be in region 4, because otherwise Z would be incompatible with both T_1 and T_2.

Now on the right, Z cannot be in regions 5 or 6, since then Z would be incompatible with T_1 or T_2, respectively. Finally, Z cannot be in region 7, since Z would not be compatible with both T_1 and T_2. Thus, if Z is not in $\mathcal{T}$ and $Z \neq Y$, then Z is not in $\mathcal{T}'$.

For (2), consider a Y' such that $(\mathcal{T} \setminus \{X\}) \cup \{Y'\}$ is a compatible set. We see that Y' cannot be in any of the labelled regions but must be incompatible with X. By Lemma 5.8 again, Y' cannot be on the interior or sides (without corners) of the tilting rectangle XT_1YT_2. Thus $Y' = Y$. $\qquad \square$

Definition 5.10. Let $\mathcal{T}$ be a **T**-cluster. Suppose $X \in \mathcal{T}$ and $Y \notin \mathcal{T}$ such that $\mathcal{T}' := (\mathcal{T} \setminus \{X\}) \cup \{Y\}$ is also a **T**-cluster. Define $\mu : \mathcal{T} \to \mathcal{T}'$ by

$$\mu(T) = \begin{cases} Y & T = X \\ T & \text{otherwise,} \end{cases}$$

and call it the **T**-*mutation* of $\mathcal{T}$ at X.

Remark 5.11. Proposition 5.9 asserts that the **T**-clusters and **T**-mutations yield a cluster theory in the sense of [IRT20, Definition 5.1.1]. The primary difference between a cluster structure as in

[BI+09] and a cluster theory in [IRT20] is that, for a cluster theory, we do not require that every object be mutable. The uniqueness of the mutation, if it exists, is required by both cluster structures and cluster theories.

Example 5.12. Suppose $\mathcal{Z}$ has at least 3 line segments, and let $\mathcal{T} = \mathcal{Z}$. Also, let ℓ_i be a line segment of $\mathcal{Z}$ and X be an interior point of ℓ_i. Then any nondegenerate rectangle $\lozenge$ in $\mathcal{C}_{\mathcal{Z}}$ whose left corner is X has a side that intersects $\mathcal{T} = \mathcal{Z}$ at infinitely many points. By Lemma 5.8, there is no **T**-mutation $\mathcal{T} \to \mathcal{T}'$ at X with $\mathcal{T}' \neq \mathcal{T}$.
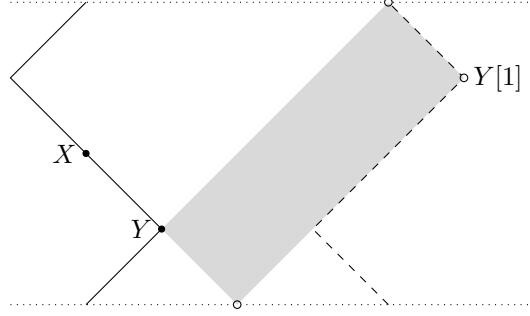
In contrast, choose Y to be the intersection of ℓ_i and ℓ_{i+1}, where ℓ_i has slope -1. (By our assumption on $\mathcal{Z}$, there exists such a Y.) Then, we can find a **T**-mutation of $\mathcal{T}$ at Y, given by $\mathcal{T}' := (\mathcal{T} \setminus \{Y\}) \cup \{Y[1]\}$. For a given zigzag $\mathcal{Z}$, all the possible ways to **T**-mutate $\mathcal{Z}$ are discussed in Section 6.3.

6. The associahedron

Throughout this section, $\mathcal{Z}$ denotes a zigzag in $\mathcal{D}$ (Definition 3.1), and by $\mathcal{C}_{\mathcal{Z}}$ we denote the associated full subcategory in $\mathcal{D}$ introduced in Notation 5.1. Moreover, we fix a permissible function $\underline{c} : \mathrm{Ind}(\mathcal{D}) \sqcup \{0\} \to \mathbb{R}$ (Definition 3.17) such that $\underline{c}(X) > 0$ for all indecomposable objects X in $\mathcal{C}_{\mathcal{Z}}$.

6.1. **Solutions.** As recalled in Section 2.1, in the finite setting there is a correspondence between the clusters of the associated cluster algebra and the solutions of certain systems of equations arising from the mesh relations, as discussed in [BD+18]. To employ this idea in the continuous setting, we first generalize the notion of a solution with respect to a system of linear equations induced by the continuous deformed mesh relations. Then, we connect **T**-clusters to such solutions and show that for a solution Φ and each indecomposable X in $\mathcal{C}_{\mathcal{Z}}$, the possible values of $\Phi(X)$ are bounded.

Definition 6.1. With the notations as above, a function $\Phi : \mathrm{Ind}(\mathcal{D}) \sqcup \{0\} \to \mathbb{R}$ is called a *solution with respect to* $\underline{c}$ *in* $\mathcal{C}_{\mathcal{Z}}$ if it satisfies the following conditions.

- Φ satisfies the continuous deformed mesh relations over $\underline{c}$ (Definition 3.18), and
- $\Phi(X) \geq 0$ for each indecomposable X in $\mathcal{C}_{\mathcal{Z}}$.

Since our ultimate goal is to relate these solutions to **T**-clusters, we are only interested in $\Phi|_{\mathrm{Ind}(\mathcal{C}_{\mathcal{Z}})}$. However, by Proposition 3.24, one can extend any such restricted function to all of $\mathrm{Ind}(\mathcal{D}) \sqcup \{0\}$. We note that this extension is unique, thus two solutions Φ_1 and Φ_2 are the same if and only if they are the same on $\mathcal{C}_{\mathcal{Z}}$.

Proposition 6.2. *Let* $\Phi : \mathrm{Ind}(\mathcal{D}) \sqcup \{0\} \to \mathbb{R}$ *be a solution with respect to* $\underline{c}$ *in* $\mathcal{C}_{\mathcal{Z}}$. *If there exists a* **T**-*cluster* $\mathcal{T}$ *such that* $\Phi(X) = 0$ *for all* $X \in \mathcal{T}$, *then* $\Phi(X) > 0$ *for all* $X \notin \mathcal{T}$.

Proof. Let W be an indecomposable in $\mathcal{C}_{\mathcal{Z}}$ such that W is not in $\mathcal{T}$. Using the same notation as in Definition 2.5, there exists $X \in \mathcal{T}$ and a tilting rectangle $\lozenge = XYWZ$ or $\lozenge = WYXZ$ contained in $\mathcal{C}_{\mathcal{Z}}$. In particular, since Y and Z are respectively the top and bottom vertices in either rectangle, and because Φ satisfies the continuous deformed mesh relations over $\underline{c}$, we have

$$\Phi(X) + \Phi(W) = \Phi(Y) + \Phi(Z) + \int_{\lozenge} \underline{c}.$$

The right side of the equation is positive since $\underline{c}$ takes positive values and $\Diamond$ is nondegenerate. $\qquad\square$

Recall that any zigzag contained in $\mathcal{C}_{\mathcal{Z}}$ is a **T**-cluster (Example 5.5). By Proposition 3.24, this means that all such clusters have a unique solution with respect to $\underline{c}$ in $\mathcal{C}_{\mathcal{Z}}$. The study of arbitrary **T**-clusters is more complex. In particular, there are totally disconnected **T**-clusters, such as the example in Section 5.2.1. In $\mathcal{T}_1$ (Figure 8) from the same section, one may instead fill the smaller triangles on the bottom with a scaled down version of any cluster that does not intersect the top point of the triangle or the left side. Furthermore, if one begins with a different zigzag, perhaps with many line segments, there are even more possible constructions.

Question 6.3. For each **T**-cluster $\mathcal{T}$, does there exist a solution Φ with respect to $\underline{c}$ in $\mathcal{C}_{\mathcal{Z}}$ such that $\Phi(X) = 0$ for all $X \in \mathcal{T}$? For which **T**-clusters is such a solution unique (when it exists)?

Let $g_{\mathcal{Z}}$-vectors (Definition 4.1) and the rays $\mathfrak{u}$ and $\mathfrak{d}$ be as defined in Section 4.2. Let X be an indecomposable object in $\mathcal{C}_{\mathcal{Z}}$ which is neither in $\mathcal{Z}$ nor in $\mathcal{Z}[1]$. Suppose GX is the region in $\mathcal{C}_{\mathcal{Z}}$ bounded by $\mathcal{Z}$, $\mathfrak{u}$, and $\mathfrak{d}$. The reader may refer back to Figure 7 for a depiction of such a GX.

Theorem 6.4. *Let X be an indecomposable object in $\mathcal{C}_{\mathcal{Z}}$. There exists a bound $\xi_X \in \mathbb{R}_{>0}$ such that $\Phi(X) \leq \xi_X$, for all solutions Φ with respect to $\underline{c}$ in $\mathcal{C}_{\mathcal{Z}}$.*

Proof. Let Φ be a solution with respect to $\underline{c}$ in $\mathcal{C}_{\mathcal{Z}}$. We first show that the possible values of the objects in $\mathcal{Z}$ and $\mathcal{Z}[1]$ are bounded.

Let $X \in \mathcal{Z}$. Let $\Diamond$ be the tilting rectangle in $\mathcal{C}_{\mathcal{Z}}$ whose left and right corners are X and $X[1]$, respectively. Note that $\Phi(X[1]) \geq 0$, and therefore the maximum possible value of $\Phi(X)$ is $\int_{\Diamond} \underline{c}$. Similarly, $\Phi(X[1]) \leq \int_{\Diamond} \underline{c}$.

Now let X be an indecomposable not in $\mathcal{Z}$ and not in $\mathcal{Z}[1]$. Let $Z_0, \ldots, Z_n$ be the objects in $\mathcal{Z}$ corresponding to the nonnegative coordinates of $g_{\mathcal{Z}}(X)$. Using a similar argument to that in the proof of Proposition 4.2, we see that

$$\Phi(X) = \begin{cases} \displaystyle\sum_{\text{odd } i} \Phi(Z_i) - \sum_{\text{even } i} \Phi(Z_i) + GX & Z_0 \text{ is left of } Z_1 \\ \displaystyle\sum_{\text{even } i} \Phi(Z_i) - \sum_{\text{odd } i} \Phi(Z_i) + GX & Z_0 \text{ is right of } Z_1. \end{cases}$$

Suppose Z_0 is left of Z_1. The other case is similar. Let $\Diamond_i$ be the rectangle in $\mathcal{C}_{\mathcal{Z}}$ whose left and right corners are Z_i and $Z_i[1]$, respectively. By using the minimum values of the even $\Phi(Z_i)$'s and the maximum values of the odd $\Phi(Z_i)$'s, it follows that

$$\Phi(X) \leq GX + \sum_{\text{odd } i} \left(\int_{\Diamond_i} \underline{c} \right).$$

Now, choose ξ_X to be the right side of the displayed inequality. This completes the proof. $\qquad\square$

6.2. **Associahedron.** Now we are equipped with the required tools to describe the titular object of the paper: a *continuous associahedron*. First, let us fix the ambient space where this associahedron will be realized. By $\prod_{\text{Ind}(\mathcal{C}_{\mathcal{Z}})} \mathbb{R}$, we denote the real vector space whose coordinates are indexed by indecomposable objects in $\mathcal{C}_{\mathcal{Z}}$.

In this subsection, we introduce the continuous associahedron and show that it is a convex object in the ambient space $\prod_{\text{Ind}(\mathcal{C}_{\mathcal{Z}})} \mathbb{R}$. Further, we prove that the solutions corresponding to **T**-clusters are extremal points (that is, on the boundary). To each function $\Phi : \text{Ind}(\mathcal{D}) \sqcup \{0\} \to \mathbb{R}$, one can naturally associate a vector in $\prod_{\text{Ind}(\mathcal{C}_{\mathcal{Z}})} \mathbb{R}$. In particular, for a fixed $\underline{c}$, we are interested in those vectors corresponding to the solutions with respect to $\underline{c}$ in $\mathcal{C}_{\mathcal{Z}}$.

Definition 6.5. For each fixed $\underline{c}$, the *continuous associahedron* $\mathbb{U}_{\mathcal{Z},\underline{c}}$ is the subset of $\prod_{\text{Ind}(\mathcal{C}_{\mathcal{Z}})} \mathbb{R}$ consisting of those vectors corresponding to the solutions with respect to $\underline{c}$ in $\mathcal{C}_{\mathcal{Z}}$.

Remark 6.6. By Proposition 3.24, each solution Φ with respect to $\underline{c}$ in $\mathcal{C}_{\mathcal{Z}}$ is determined by the values Φ takes on any zigzag in $\mathcal{C}_{\mathcal{Z}}$. In particular, if we know the values of Φ on $\mathcal{Z}[1]$ then we may recover the values of Φ on the rest of $\mathcal{C}_{\mathcal{Z}}$. So, one may consider the projection of $\mathbb{U}_{\mathcal{Z},\underline{c}}$ onto $\prod_{\mathrm{Ind}(\mathcal{Z}[1])} \mathbb{R}$ without losing any information.

As recalled in Section 2.1, in [BD+18] the authors give another characterization of $\mathbb{A}_{\underline{c}}$ in terms of the g-vectors of indecomposable modules associated to elements of $\mathcal{I}$. (See Theorem 2.3 and the paragraphs preceding it.) That is to say, $\mathbb{A}_{\underline{c}}$ is obtained as the projection of $\mathbb{U}_{\underline{c}}$ via a map determined by the g-vectors of those indecomposable objects associated to elements of $\mathcal{I}^{[1]}$. Consequently, in the setting of Dynkin quivers, they also recover $\mathbb{A}_{\underline{c}}$ as a polytopal realization of the g-vector fans. This approach has been further studied and developed in [PP+19]. Inspired by such results, it is natural to hope for an analogous realization of the associahedron in the continuous setting. Namely, to study the projection of $\mathbb{U}_{\mathcal{Z},\underline{c}}$ onto $\prod_{\mathrm{Ind}(\mathcal{Z}[1])} \mathbb{R}$ determined by the $g_{\mathcal{Z}}$-vectors corresponding to each Z in $\mathrm{Ind}(\mathcal{Z}[1])$. This direction of work requires further investigation.

We recall that a subset $\mathbb{X}$ of a vector space $\mathbb{V}$ is said to be *convex* if for each pair of points $A, B \in \mathbb{X}$ and every $t \in [0, 1]$, the linear combination $tA + (1 - t)B$ is in $\mathbb{X}$.

Theorem 6.7. *The set $\mathbb{U}_{\mathcal{Z},\underline{c}}$ is convex in $\prod_{\mathrm{Ind}(\mathcal{C}_{\mathcal{Z}})} \mathbb{R}$.*

Proof. Let Φ_0 and Φ_1 be solutions to $\underline{c}$ in $\mathcal{C}_{\mathcal{Z}}$. For each $t \in (0, 1)$, define Φ_t as

$$\Phi_t(X) := t \cdot \Phi_1(X) + (1 - t) \cdot \Phi_0(X).$$

For each tilting rectangle $\Diamond = XYWZ$ in $\mathcal{C}_{\mathcal{Z}}$, we have the following equations from Φ_0 and Φ_1:

$$\Phi_0(X) + \Phi_0(W) = \Phi_0(Y) + \Phi_0(Z) + \int_{\Diamond} \underline{c}$$

$$\Phi_1(X) + \Phi_1(W) = \Phi_1(Y) + \Phi_1(Z) + \int_{\Diamond} \underline{c}.$$

Since $\int_{\Diamond} \underline{c}$ is fixed, we have

$$\Phi_t(X) + \Phi_t(W) = \Phi_t(Y) + \Phi_t(Z) + \int_{\Diamond} \underline{c}.$$

Thus, any line segment in $\prod_{\mathrm{Ind}(\mathcal{C}_{\mathcal{Z}})} \mathbb{R}$ connecting two points in $\mathbb{U}_{\mathcal{Z},\underline{c}}$ is entirely contained in $\mathbb{U}_{\mathcal{Z},\underline{c}}$. $\quad\square$

The previous theorem and its proof allow us to make the following definition.

Definition 6.8. Let $\Phi \in \mathbb{U}_{\mathcal{Z},\underline{c}}$ be a solution with respect to $\underline{c}$ in $\mathcal{C}_{\mathcal{Z}}$. We say Φ is *on the boundary* of $\mathbb{U}_{\mathcal{Z},\underline{c}}$ if there exists a line segment ℓ parameterized by $t \in [0, 1]$ satisfying the following conditions.

- $\Phi = \Phi_1$ and $\Phi_0 \in \mathbb{U}_{\mathcal{Z},\underline{c}}$ are the distinct endpoints of ℓ (and so $\ell \subsetneq \mathbb{U}_{\mathcal{Z},\underline{c}}$).
- If ℓ' is a parameterized line segment that contains ℓ such that Φ_1 is not an endpoint of ℓ', then ℓ' is not contained in $\mathbb{U}_{\mathcal{Z},\underline{c}}$.

We call the set of all such Φ the *boundary* of $\mathbb{U}_{\mathcal{Z},\underline{c}}$.

Theorem 6.9. *Let $\mathcal{T}$ be a **T**-cluster and Φ a solution with respect to $\underline{c}$ in $\mathcal{C}_{\mathcal{Z}}$ such that $\Phi(X) = 0$ for all $X \in \mathcal{T}$. Then Φ is on the boundary of $\mathbb{U}_{\mathcal{Z},\underline{c}}$.*

Proof. Let $X \in \mathcal{T}$, $\Phi_1 = \Phi$, and $\mathcal{T}'$ be any zigzag contained in $\mathcal{C}_{\mathcal{Z}}$ that does not contain X. By Proposition 3.24, there is a unique solution Φ_0 such that $\Phi_0(X') = 0$ for all $X' \in \mathcal{T}'$. For all $t \in \mathbb{R}$ and indecomposables X in $\mathcal{C}_{\mathcal{Z}}$, let

$$\Phi_t(X) := t \cdot \Phi_1(X) + (1 - t) \cdot \Phi_0(X).$$

Let $\varepsilon > 0$ and note $\Phi_{1+\varepsilon}(X) = -\varepsilon \cdot \Phi_0(X)$. Since $X \notin \mathcal{T}'$, Proposition 6.2 asserts that $\Phi_0(X) > 0$. Thus, $\Phi_{1+\varepsilon}(X) < 0$ and so $\Phi_{1+\varepsilon} \notin \mathbb{U}_{\mathcal{Z},\underline{c}}$. Therefore, $\Phi = \Phi_1$ is on the boundary of $\mathbb{U}_{\mathcal{Z},\underline{c}}$. $\quad\square$

6.3. **T-mutation a continuous associahedron.** Recall that in the finite setting (Section 2.1), the vertices of the ABHY associahedron correspond to the clusters, the facets are in bijection with cluster variables, and the mutation of clusters corresponds to the edges which connect two vertices of the associahedron. These edges are given by the intersection of the hyperplanes associated to the mutable cluster variables. (For full details, see [BD+18].)

In this subsection, we discuss analogous phenomena in the continuous setting. In particular, we describe the connections between **T**-mutations and the continuous associahedron $\mathbb{U}_{\mathcal{Z},\underline{c}}$. Namely, we show that for two different **T**-clusters, each of which corresponds to a unique solution in $\mathbb{U}_{\mathcal{Z},\underline{c}}$, the **T**-mutation can be seen via $\mathbb{U}_{\mathcal{Z},\underline{c}}$.

Notation 6.10. For an indecomposable object X in $\mathcal{C}_{\mathcal{Z}}$, by $\mathbb{H}_{\mathcal{Z},\underline{c}}(X)$ we denote the *hyperplane* in $\prod_{\mathrm{Ind}(\mathcal{C}_{\mathcal{Z}})} \mathbb{R}$ associated to X, which is the set of functions $\Phi : \mathrm{Ind}(\mathcal{C}_{\mathcal{Z}}) \sqcup \{0\} \to \mathbb{R}$ such that $\Phi(X) = 0$.

Theorem 6.11. *Let $\mathcal{T}$ and $\mathcal{T}'$ be **T**-clusters with respective unique solutions Φ and Φ' in $\mathbb{U}_{\mathcal{Z},\underline{c}}$. The following are equivalent.*

(1) *There is a **T**-mutation $\mu : \mathcal{T} \to \mathcal{T}'$, where $\mathcal{T}' = (\mathcal{T} \setminus \{T_0\}) \cup \{T_1\}$, with $T_0 \in \mathcal{T}$ and $T_1 \in \mathcal{T}'$.*
(2) *There is a line segment $\{\Phi_t \mid t \in [0,1]\}$ which connects Φ to Φ' in $\mathbb{U}_{\mathcal{Z},\underline{c}}$ such that $\Phi_0 = \Phi$, $\Phi_1 = \Phi'$, and $\Phi_t(X) = 0$, for all $X \in \mathcal{T} \cap \mathcal{T}'$.*

The line segment in the second part is given by

$$\left(\bigcap_{X \in \mathcal{T} \cap \mathcal{T}'} \mathbb{H}_{\mathcal{Z},\underline{c}}(X) \right) \cap \mathbb{U}_{\mathcal{Z},\underline{c}},$$

where $\mathcal{T} \cap \mathcal{T}' = \mathcal{T} \setminus \{T_0\} = \mathcal{T}' \setminus \{T_1\}$.

Proof. If (2) holds, to get (1) apply Proposition 5.9 to the equation $\mathcal{T} \cap \mathcal{T}' = \mathcal{T} \setminus \{T_0\} = \mathcal{T}' \setminus \{T_1\}$.

Now assume (1). Let $\Phi_0 = \Phi$ and $\Phi_1 = \Phi'$. Moreover, for all indecomposable X in $\mathcal{C}_{\mathcal{Z}}$ define

$$\Phi_t(X) = t \cdot \Phi_1(X) + (1 - t) \cdot \Phi_0(X),$$

for $t \in [0, 1]$. Then, for any X in $\mathcal{T} \cap \mathcal{T}'$ we have $\Phi_t(X) = 0$. Since $\mathbb{U}_{\mathcal{Z},\underline{c}}$ is convex (Theorem 6.7), we have the desired line segment. We know Φ and Φ' are unique, so we have the singleton sets:

$$\{\Phi\} = \left(\bigcap_{X \in \mathcal{T}} \mathbb{H}_{\mathcal{Z},\underline{c}}(X) \right) \cap \mathbb{U}_{\mathcal{Z},\underline{c}} \qquad \text{and} \qquad \{\Phi'\} = \left(\bigcap_{X \in \mathcal{T}'} \mathbb{H}_{\mathcal{Z},\underline{c}}(X) \right) \cap \mathbb{U}_{\mathcal{Z},\underline{c}}.$$

By Proposition 5.9, the line segment is the desired intersection. $\square$

For the next proposition, recall the definition of a right vertex (Definition 3.1). An example of the setting and statement of the proposition can be seen in Figure 10.

Proposition 6.12. *Suppose $\mathcal{Z}$ has n right vertices, and let $\mathscr{M}_{\mathcal{Z}}$ be the set of **T**-clusters obtained from $\mathcal{Z}$ via finitely many **T**-mutations. Then $\mathscr{M}_{\mathcal{Z}}$ and the edges in $\mathbb{U}_{\mathcal{Z},\underline{c}}$ corresponding to the **T**-mutations form the 1-skeleton of an n-dimensional hypercube in $\mathbb{U}_{\mathcal{Z},\underline{c}}$. In particular, $\mathscr{M}_{\mathcal{Z}}$ is finite.*

Proof. We first note that the only **T**-mutable objects in $\mathcal{Z}$ are the right vertices (see Example 5.12). For a vertex X, consider the mutation $\mathcal{Z} \to (\mathcal{Z} \setminus \{X\}) \cup \{X[1]\}$. Choose a different right vertex Y of $\mathcal{Z}$. We may also mutate

$$(\mathcal{Z} \setminus \{X\}) \cup \{X[1]\} \to (((\mathcal{Z} \setminus \{X\}) \cup \{X[1]\}) \setminus \{Y\}) \cup \{Y[1]\}.$$

We note that the mutation of X and Y are independent of each other. Further, observe that X and Y were chosen arbitrarily, therefore this argument holds for all n right vertices of $\mathcal{Z}$.

We then have a bijection from the set of **T**-clusters that can be obtained from $\mathcal{Z}$ in finitely many **T**-mutations to the set $\{0,1\}^n$. This is in particular the number of vertices of an n-dimensional hypercube. The commutativity of the mutations yields the n-dimensional hypercube structure. $\square$
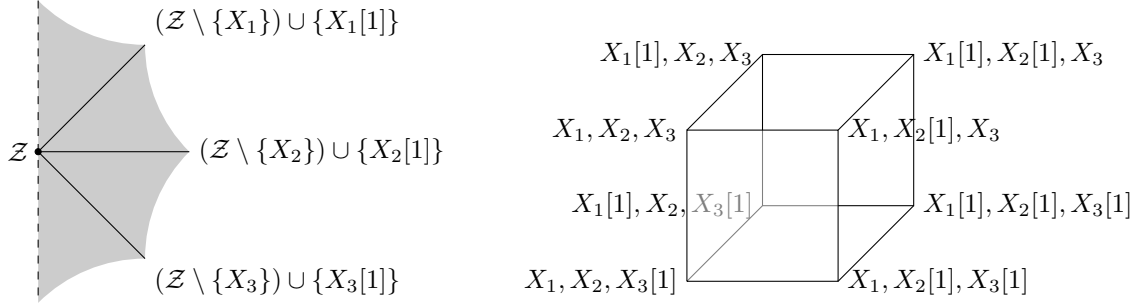
FIGURE 10. We illustrate possible mutations where $\mathcal{T} = \mathcal{Z}$ has 3 right vertices: X_1, X_2, X_3 (Definition 3.1). On the left, we have the web-like effect for mutation at the unique Φ in $\mathbb{U}_{\mathcal{Z},\underline{c}}$ such that $\Phi(X) = 0$ for all $X \in \mathcal{Z}$. The vertices represented by the webbing between the lines are not **T**-mutable. On the right, we have the mutation structure starting with a zigzag $\mathcal{Z}$ with 3 right vertices. Each facet corresponds to one of X_1, X_2, X_3, $X_1[1]$, $X_2[1]$, and $X_3[1]$, and the vertices are labelled by the facets they belong to. Thus, the vertices are the corresponding **T**-clusters.

6.4. **Finite embeddings.** The main goal of this section is to study the relationship between the cluster structures of type A_n and the **T**-clusters. In particular, in Subsection 6.4.1 we return to Example 5.5 (2) to prove a technical lemma (Lemma 6.13) that we need to deduce Theorem 6.18. In Subsection 6.4.2, we use the results of the first subsection to complete our argument.

6.4.1. *Using Example 5.5 (2).* We now show that each **T**-cluster $\mathcal{T}$ as in Example 5.5 (2) has a unique solution Φ in $\mathbb{U}_{\mathcal{Z},\underline{c}}$ such that $\Phi(X) = 0$ for all $X \in \mathcal{T}$.

Let ℓ be a curve between $\mathcal{Z}$ and $\mathcal{Z}[1]$ such that the slope of ℓ at each point is greater than 1, less than -1, or equal to ∞. Additionally, suppose that for all $a \in (-\frac{\pi}{2}, \frac{\pi}{2})$, there is an $A \in \ell$ such that the y-coordinate of A is a. Assign $\Phi(A) = 0$ for all A on ℓ.

Now we construct a tilting rectangle that contains part of ℓ. It may be helpful for the reader to refer to Figure 11 while reading this construction. Let X be an indecomposable in $\mathcal{C}_{\mathcal{Z}}$ such that X is not on ℓ and X is to the left of ℓ. Without loss of generality, we assume that the rays emanating from X at $\pm 45°$ intersect ℓ. Due to our conditions on ℓ, there is a unique X' not on ℓ such that the tilting rectangle with left and right corners X and X' has top and bottom corners on ℓ. We create a sequence of regions $R_1 \subsetneq R_2 \subsetneq \cdots$ such that

$$\Phi(X) = \int_{\lim_{n \to \infty} R_n} \underline{c}.$$

We now choose X such that X and X' above are both in $\mathcal{C}_{\mathcal{Z}}$. Let $\Diamond$ be the tilting rectangle whose left and right corners are X and X', respectively, and whose top and bottom corners are on ℓ. Then

$$\Phi(X) + \Phi(X') = \int_{\Diamond} \underline{c}.$$

We subdivide the rectangle with objects Y, Y', Z, Z', each distinct from each other and from X, X', such that each of the four smaller rectangles share a corner on ℓ (illustrated in Figure 11). Let R_1 be the tilting rectangle whose left corner is X, top corner is Y, bottom corner is Z, and right corner is on ℓ. Then we have

$$\Phi(X) = \Phi(Y) + \Phi(Z) + \int_{R_1} \underline{c}.$$

In order for Φ to be a solution with respect to $\underline{c}$ in $\mathbb{U}_{\mathcal{Z},\underline{c}}$, we must have $\Phi(X) \geq 0$, $\Phi(Y) \geq 0$, and $\Phi(Z) \geq 0$. Thus, we must have $\Phi(X) \geq \int_{R_1} \underline{c}$. By a similar process, we create two smaller tilting rectangles whose left corners are Y and Z, respectively, and whose right corners are on ℓ. Let R_2 be
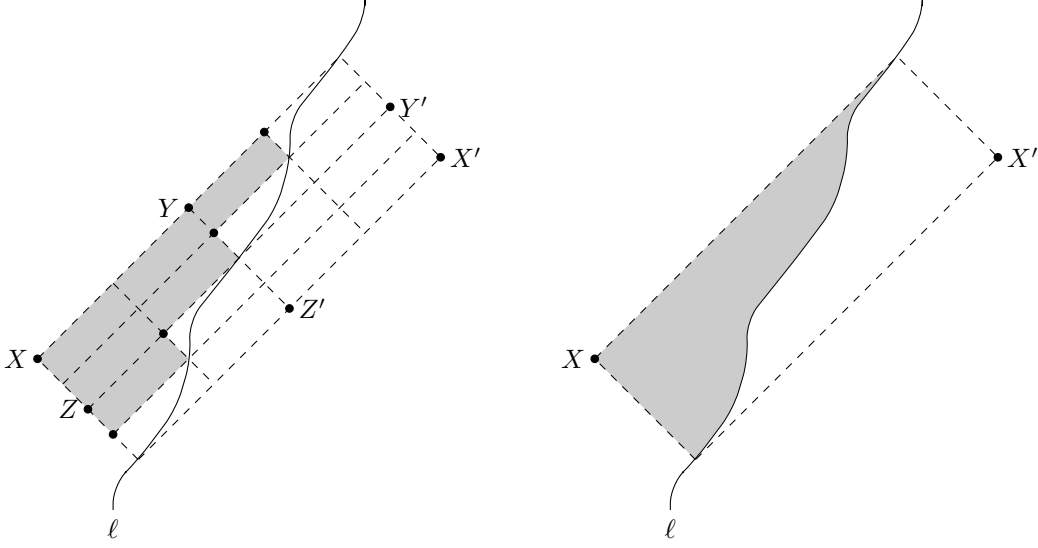
FIGURE 11. On the left, we have the region R_2 obtained in the processs of finding a lower bound of $\Phi(X)$, where Φ is 0 on ℓ. On the right, we have the region R such that $\Phi(X) = \int_R \underline{c}$.

the union of R_1 and the two smaller rectangles. If we want to extend Φ to a solution with respect to $\underline{c}$ in $\mathcal{C}_{\mathcal{Z}}$, we must have $\Phi(X) \geq \int_{R_2} \underline{c}$.

We continue defining successively larger regions $R_3 \subsetneq R_4 \subsetneq \cdots$ similarly. The limit $R = \lim_{n \to \infty} R_n$ is the region inside $\Diamond$ on the left of ℓ, and we see $\Phi(X) \geq \int_R \underline{c}$. Denote by R' the region in $\Diamond$ on the right of ℓ so that $R \cup R' \cup \ell = \Diamond$. Then $\Phi(X') \geq \int_{R'} \underline{c}$ by the same argument. Since $\Phi(X) + \Phi(X') = \int_{\Diamond} \underline{c}$, we must have $\Phi(X) = \int_R \underline{c}$ and $\Phi(X') = \int_{R'} \underline{c}$. Repeating this argument for each point in R shows that Φ extends uniquely to all of $\Diamond$.

Lemma 6.13. *Let ℓ be a curve in $\mathcal{C}_{\mathcal{Z}}$ as in Example 5.5 (2). Then there is a unique solution Φ with respect to $\underline{c}$ in $\mathcal{C}_{\mathcal{Z}}$ such that $\Phi(A) = 0$, for all $A \in \ell$.*

Proof. Note that $\ell \subsetneq \mathrm{Ind}(\mathcal{C}_{\mathcal{Z}})$. Since $\mathcal{Z}$ and $\mathcal{Z}[1]$ have finitely many line segments, we find finitely many tilting rectangles $\{\Diamond_i\}_{i=1}^n$ in $\mathrm{Ind}(\mathcal{C}_{\mathcal{Z}})$ with the following three properties. First, $\ell \subsetneq \bigcup_{i=1}^n \Diamond_i$. Second, for $1 \leq i < n$, the bottom corner of $\Diamond_i$ is the top corner of $\Diamond_{i+1}$. Third, top corner of $\Diamond_1$ and bottom corner of $\Diamond_n$ have y-coordinates $\frac{\pi}{2}$ and $-\frac{\pi}{2}$, respectively. By the construction preceding the lemma, we extend Φ uniquely on each of these $\Diamond_i$'s. The left sides of all $\Diamond_i$'s form a zigzag in $\mathrm{Ind}(\mathcal{C}_{\mathcal{Z}})$. By Proposition 3.24, there is a unique extension of Φ as stated in the proposition. $\qquad\square$

6.4.2. *Construction of the embeddings.* In this subsection we employ our argument in Section 6.4.1 to prove Theorem 6.18. Throughout the section, we use our notation from Section 2.1. For the theorem, we consider the ascending linear orientation of A_n, for $n \geq 2$, meaning that the A_n quiver is linearly ordered and $P_i \hookrightarrow P_j$ if $i > j$. There is an exact embedding $\mathrm{rep}(A_2) \to \mathrm{rep}(A_3)$ determined by sending $P_1 \mapsto P_1$ and $P_2 \mapsto P_2$. In general, there is an exact embedding $\mathrm{rep}(A_n) \to \mathrm{rep}(A_{n+1})$ by sending $P_i \mapsto P_i$, for $1 \leq i \leq n$. Also, for simplicity, we assume the zigzag $\mathcal{Z}$ consists of exactly one line segment, which has slope $+1$. Otherwise, the computations become exceedingly involved.

We construct a **T**-cluster $\mathcal{T}_U$ (Figure 12) similar to $\mathcal{T}_1$ from Section 5.2.1 (see Figure 8). We consider $\{X_i = (x_i, y_i) \mid i \in \mathbb{Z}_{<0}\}$, as a sequence of indecomposables in $\mathcal{Z}$ such that $j < i$ implies $y_j < y_i$ and $\lim_{j \to -\infty} y_j = -\frac{\pi}{2}$. Let $(x_0, \frac{\pi}{2})$ be the top boundary point of $\mathcal{Z}$. For $i < 0$, we define
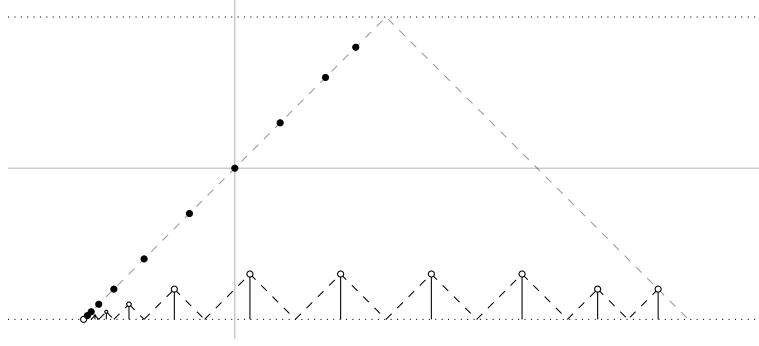
FIGURE 12. The **T**-cluster $\mathcal{T}_U$ used to construct the finite embeddings.

E_i as in Section 5.2.1 and a vertical line segment ℓ_i:

$$E_i := \left(x_{i+1} + (x_i + \frac{\pi}{2}),\, y_{i+1} - (y_i + \frac{\pi}{2}) \right),$$

$$\ell_i := \left\{ (x, y) \mid x = x_{i+1} + (x_i + \frac{\pi}{2}),\, y < y_{i+1} - (y_i + \frac{\pi}{2}) \right\}.$$

Note that ℓ_i does not include E_i. Let

$$\mathcal{T}_U = \{ X_i \mid i < 0 \} \cup \left(\bigcup_{i<0} \ell_i \right).$$

As in $\mathcal{T}_1$, we have the smaller triangles along the bottom. However, $\mathcal{T}_U$ has a rightmost triangle on the bottom. One may verify that $\mathcal{T}_U$ is a **T**-cluster.

By Lemma 5.8, only the discrete set of points on $\mathcal{Z}$ are **T**-mutable. As in Section 4.1, $\mathcal{D}^\heartsuit$ denotes the heart of the t-structure induced by $\mathcal{Z}$ in the category $\mathcal{D}$.

Proposition 6.14. *For $n \geq 2$, there is an exact embedding $\mathrm{rep}(A_n) \to \mathcal{D}^\heartsuit$ of abelian categories determined by $P_{-i} \mapsto X_i$, for $-n \leq i \leq -1$. The embedding factors as $\mathrm{rep}(A_n) \to \mathrm{rep}(A_{n+1}) \to \mathcal{D}^\heartsuit$.*

Proof. From the paragraph preceding the proposition, it follows that all pairwise distinct indecomposable projective objects in $\mathrm{rep}(A_n)$ are sent to pairwise distinct indecomposable projective objects in $\mathcal{D}^\heartsuit$. Furthermore, for any two indecomposables E, F in $\mathrm{rep}(A_n)$, the space of morphisms from E to F in $\mathrm{rep}(A_n)$ is isomorphic to the space of morphisms of the corresponding indecomposables in $\mathcal{D}^\heartsuit$. From this, one may check that the embedding is exact. The factorization follows from the definitions of the embeddings. $\square$

The indecomposable objects of $\mathcal{D}^b(A_n)$ and $\mathcal{D}$ are shifts of copies of $\mathrm{rep}(A_n)$ and $\mathcal{D}^\heartsuit$, respectively. The following proposition then follows from straightforward computations.

Proposition 6.15. *With the same notation as above, for each $n \geq 2$, there is a triangulated embedding $\Theta_n : \mathcal{D}^b(A_n) \to \mathcal{D}$ determined by sending $P_{-i}[m] \to X_i[m]$, for all $m \in \mathbb{Z}$. Furthermore, the embedding factors as $\mathcal{D}^b(A_n) \to \mathcal{D}^b(A_{n+1}) \to \mathcal{D}$.*

Consider again the **T**-cluster $\mathcal{T}_U$ defined at the beginning of this section (depicted in Figure 12). The following notation will be of use in the proof of Theorem 6.18. Recall $\mathcal{I}^+$ as defined in Section 2.1, page 5.

Notation 6.16. Let $\mathcal{S}$ be a cluster of type A_n consisting of indecomposables in $\mathrm{add}(\mathcal{I}^+)$. Using Θ_n in Proposition 6.15, set $\widetilde{\mathcal{S}} := (\mathcal{T}_U \setminus \{X_i\}_{i \geq -n}) \cup \Theta(\mathcal{S})$, where $\{X_i \mid i \in \mathbb{Z}_{<0}\}$ is as before.

Proposition 6.17. *With the same notation $\mathcal{T}_U$ as above, there is a unique solution Φ with respect to $\underline{c}$ in $\mathcal{C}_{\mathcal{Z}}$ such that $\Phi(X) = 0$, for all $X \in \mathcal{T}_U$.*

Proof. Let $\Phi(X) = 0$ for all $X \in \mathcal{T}_U$. Then Φ is defined on a countable number of points on $\mathcal{Z}$, but not on all of $\mathcal{Z}$. By Lemma 6.13, if we consider each of the small triangles in the bottom of Figure 8 as a triangular patch, this Φ uniquely extends to all of them. From here it is straightforward to check that we may uniquely extend Φ to all of $\mathcal{Z}$. Then, Proposition 3.24 implies that Φ uniquely extends to all of $\mathcal{C}_\mathcal{Z}$. Thus, we have a unique Φ as in the statement of the proposition. $\square$

For each A_n, by Proposition 6.15, we may consider $\mathrm{add}(\mathcal{I}^+)$ as a subcategory of $\mathcal{C}_\mathcal{Z}$. The almost split triangles in $\mathrm{add}(\mathcal{I}^+)$, inherited from $\mathcal{D}^b(A_n)$, determine tilting rectangles in $\mathcal{C}_\mathcal{Z}$. Assign to each indecomposable E in $\mathrm{rep}(A_n)$ the value of $\int \underline{c}$ over the corresponding tilting rectangle in $\mathcal{C}_\mathcal{Z}$ determined by the almost split triangle starting with E. While not integer values, we may still find values for the deformed mesh relations in $\mathrm{add}(\mathcal{I}^+)$. Denote by $\mathbb{U}_{n,\underline{c}}$ the associahedron obtained from A_n using $\underline{c}$ in this way.

Theorem 6.18. *There is an infinite sequence of embeddings*

$$\mathbb{U}_{2,\underline{c}} \hookrightarrow \mathbb{U}_{3,\underline{c}} \hookrightarrow \cdots \hookrightarrow \mathbb{U}_{n,\underline{c}} \hookrightarrow \mathbb{U}_{n+1,\underline{c}} \hookrightarrow \cdots \mathbb{U}_{\mathcal{Z},\underline{c}}.$$

For $n \geq 2$, the composition of embeddings $\mathbb{U}_{n,\underline{c}} \to \mathbb{U}_{\mathcal{Z},\underline{c}}$ takes the point corresponding to any cluster $\mathcal{S}$ to the unique solution Φ with respect to $\underline{c}$ in $\mathcal{C}_\mathcal{Z}$ such that $\Phi(X) = 0$ for all $X \in \widetilde{\mathcal{S}}$ (Notation 6.16). Furthermore, the composition $\mathbb{U}_{n,\underline{c}} \to \mathbb{U}_{\mathcal{Z},\underline{c}}$ takes a mutation edge to a $\mathbf{T}$-mutation edge.

Proof. We first show that each $\Theta_n|_{\mathrm{add}(\mathcal{I}^+)}$ in Proposition 6.15 takes clusters in $\mathrm{add}(\mathcal{I}^+)$ to $\mathbf{T}$-clusters in $\mathcal{C}_\mathcal{Z}$. Then we show that $\Theta_n|_{\mathrm{add}(\mathcal{I}^+)}$ takes mutations to $\mathbf{T}$-mutations.

Let $\mathcal{S}$ be a cluster in $\mathrm{add}(\mathcal{I}^+)$ and $\widetilde{\mathcal{S}}$ be as in Notation 6.16. Suppose $Y \in \mathcal{C}_\mathcal{Z}$ but $Y \notin \widetilde{\mathcal{S}}$. We show that there must exist a $T \in \widetilde{\mathcal{S}}$ such that T and Y are incompatible. If there exists $T \in \ell_i$, for some $i < 0$, such that T and Y are incompatible we are done. So, for every $i < 0$, suppose Y is compatible with all $X \in \ell_i$. Using Figure 12, we see the rays extending from Y in the negative y-direction with slope -1 and $+1$ must intersect some E_j and E_i, respectively. Otherwise, one of the rays intersects one of the ℓ_i's, which contradicts our assumption. This is because we took A_n to be the linearly ordered quiver with $P_i \hookrightarrow P_j$ if $i > j$.

If $E_i = E_{-1}$ then $Y = X_j[1]$ because $Y \in \mathrm{Ind}(\mathcal{Z}[1])$, and there is a distinguished triangle $X_j \to 0 \to Y \xrightarrow{\cong} X_j[1]$. If $E_i \neq E_{-1}$, there is a distinguished triangle $X_j \to X_{i+1} \to Y \to$ in $\mathcal{D}$ with all terms in $\mathcal{C}_\mathcal{Z}$. In either case, we use the fact that Θ_n is a triangulated embedding by Proposition 6.15. This implies there is a distinguished triangle $P_j \to W \to F \to$ in $\mathcal{D}^b(A_n)$ whose terms are in $\mathrm{add}(\mathcal{I}^+)$, such that $Y = \Theta_n(F)$. Since $Y \notin \widetilde{\mathcal{S}}$, we have $F \notin \mathcal{S}$. Hence, there is $S \in \mathcal{S}$ such that S and F are incompatible. Using Proposition 6.15 again, we have $\Theta_n(S)$ and Y are incompatible. Now $\Theta_n(S)$ is the desired $T \in \widetilde{\mathcal{S}}$ such that T and Y are incompatible. This shows that each $\Theta_n|_{\mathrm{add}(\mathcal{I}^+)}$ in Proposition 6.15 takes clusters in $\mathrm{add}(\mathcal{I}^+)$ to $\mathbf{T}$-clusters in $\mathcal{C}_\mathcal{Z}$.

Now, we let $\mu : \mathcal{S} \to \mathcal{S}'$ be a mutation of clusters in $\mathrm{add}(\mathcal{I}^+)$. Then $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{S}'}$ differ by one element. Thus, by Proposition 5.9, $\widetilde{\mathcal{S}} \to \widetilde{\mathcal{S}'}$ is a $\mathbf{T}$-mutation.

We finish the proof by showing that each Θ_n induces a geometric embedding $\mathbb{U}_{n,\underline{c}} \to \mathbb{U}_{\mathcal{Z},\underline{c}}$. We index the coordinates of the ambient space $\prod_{\mathcal{I}^+} \mathbb{R}$ of $\mathbb{U}_{n,\underline{c}}$ by the indecomposables in $\mathcal{I}^+$. Then $\Theta_n|_{\mathrm{add}(\mathcal{I}^+)}$ induces a geometric embedding $\prod_{\mathcal{I}^+} \mathbb{R} \hookrightarrow \prod_{\mathrm{Ind}(\mathcal{C}_\mathcal{Z})} \mathbb{R}$. Note that the deformed mesh relations in $\mathrm{add}(\mathcal{I}^+)$ (Section 2.1, Equation (2)) satisfy the continuous deformed mesh relations in $\mathcal{C}_\mathcal{Z}$ (Definition 3.18). Thus, the embedding takes nonnegative solutions to nonnegative solutions. Given our embeddings, for each $n \geq 2$ we may consider an A_n cluster and complete it to an A_{n+1} cluster by including the projective P_{n+1}. By this assumption and the paragraph before the theorem, the deformed mesh relations for A_n can be viewed as a set of deformed mesh relations for A_{n+1}.

Furthermore, for each A_n cluster $\mathcal{S}$, there exists a sequence of finitely many mutations $\{\mu_i\}_{i=1}^m$ such that $\mu_m \cdots \mu_1$ takes $\mathcal{S}$ to the cluster of projective indecomposables. Therefore, by our previous argument, the corresponding sequence of $\mathbf{T}$-mutations of $\widetilde{\mathcal{S}}$ forms a finite sequence that takes $\widetilde{\mathcal{S}}$ to $\mathcal{T}_U$. Then by Proposition 6.17, for each $\widetilde{\mathcal{S}}$ above, there is a unique solution Φ with respect to $\underline{c}$ in

$\mathcal{C}_{\mathcal{Z}}$ such that $\Phi(X) = 0$, for all $X \in \widetilde{\mathcal{S}}$. Finally, since each Θ_n takes A_n clusters to **T**-clusters and mutations to **T**-mutations, Theorem 6.11 implies the final statement of the theorem. $\qquad\square$

We remark that in Theorem 6.18 there is no $\mathbb{U}_{n,\underline{c}}$ (or indeed any discrete associahedron) that immediately precedes $\mathbb{U}_{\mathcal{Z},\underline{c}}$.

References

[AB+18] N. Arkani-Hamed, Y. Bai, S. He, G. Yan, *Scattering forms and the positive geometry of kinematics, color and the worldsheet*, Journal of High Energy Physics (2018), no. 5, DOI:10.1007/JHEP05(2018)096

[AH+19] N. Arkani-Hamed, S. He, G. Salvatori, H. Thomas, *Causal diamonds, cluster polytopes and scattering amplitudes*, arXiv:1912.12948 [hep-th] (2019), https://arxiv.org/abs/1912.12948

[ASS06] I. Assem, D. Simson, A. Skowroński, *Elements of the representation theory of associative algebras. vol. 1*, London Mathematical Society Student Texts **65** Cambridge University Press (2006), x+458, DOI:10.1017/CBO9780511614309

[BD+18] V. Bazier-Matte, G. Douville, K. Mousavand, H. Thomas, E. Yıldırım, *ABHY Associahedra and Newton polytopes of F-polynomials for finite type cluster algebras*, arXiv:1808.09986 [math.RT] (2018), https://arxiv.org/abs/1808.09986

[BI+09] A. B. Buan, O. Iyama, I. Reiten, J. Scott, *Cluster structures for 2-Calabi-Yau categories and unipotent groups*, Compositio Mathematica **145** (2009), no. 4, 1035–1079, DOI:10.1112/S0010437X09003960

[BM+06] A. B. Buan, B. Marsh, M. Reineke, I. Reiten, G. Todorov, *Tilting theory and cluster combinatorics*, Advances in Mathematics **204 (2)** (2006), no. 4, 572-618. DOI: 10.1016/J.AIM.2005.06.003

[BG+19] E. Barnard, E. Gunawan, E. Meehan, R. Schiffler, *Cambrian combinatorics on quiver representations (type A)*, arXiv:1912.02840v2 [math.RT] (2019), https://arxiv.org/pdf/1912.02840v2.pdf

[BM+06] A. B. Buan, B. Marsh, M. Reineke, I. Reiten, G. Todorov, *Tilting theory and cluster combinatorics*, Advances in Mathematics **204** (2006), no. 2, 572–618, DOI:10.1016/j.aim.2005.06.003

[CSZ15] C. Ceballos, F. Santos, G. M. Ziegler, *Many non-equivalent realizations of the associahedron*, Combinatorica **35** (2015) no. 5, 513–551, DOI:10.1007/s00493-014-2959-9

[CFZ02] F. Chapoton, S. Fomin, A. Zelevinsky, *Polytopal realizations of generalized associahedra*, Canadian Mathematical Bulletin **45** (2002), no. 4, 537–566, DOI:10.4153/CMB-2002-054-1

[FZ01] S. Fomin, A. Zelevinksy, *Y-systems and generalized associahedra*, Annals of Mathematics **158** (2001), no. 3, 977-1018, DOI:10.4007/annals.2003.158.977

[IRT19] K. Igusa, J. Rock, G. Todorov, *Continuous quivers of type A (I) Foundations*, arXiv:1909.10499 [math.RT] (2019), https://arxiv.org/abs/1909.10499

[IRT20] K. Igusa, J. Rock, G. Todorov, *Continuous quivers of type A (III) Embeddings of cluster theories*, arXiv:2004.10740 [math.RT] (2020), http://arxiv.org/abs/2004.10740

[IT15] K. Igusa, G. Todorov, *Continuous cluster categories I*, Algebras and Representation Theory **18** (2015), 65–101, DOI:10.1007/s10468-014-9481-z

[PP+19] A. Padrol, Y. Palu, V. Pilaud, P. Plamondon, *Associahedra for finite type cluster algebras and minimal relations between g-vectors*, arXiv:1906.06861 [math.RT] (2019), https://arxiv.org/abs/1906.06861

[R19] J. Rock, *Continuous quivers of type A (II): The Auslander–Reiten space*, arXiv:1910.04140 [math.RT] (2019), https://arxiv.org/abs/1910.04140

[S63] J. D. Stasheff, *Homotopy associativity of H-spaces. I, II*, Transactions of the American Mathematical Society **108** (1963), 293–312, DOI:10.1090/s0002-9947-1963-0158400-5

[T51] D. Tamari, *Monoïdes préordonnés et chaînes de Malcev*, Thèse, Université de Paris (1951), iv+81 pp. (mimeographed)

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