A combinatorial Fourier transform for quiver representation varieties in type A

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Goal

Consider the quiver $\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet.$

Notation:

1. $E(\mathbf{w})$ - space of representations for dimension vector $\mathbf{w} = (w_1, \ldots, w_n)$
2. $G(\mathbf{w}) = \mathbf{GL}(w_1) \times \cdots \times \mathbf{GL}(w_n)$
3. $\mathbf{w}^* = (w_n, \ldots, w_1)$ - the reverse dimension vector

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Can we give a combinatorial description of the Fourier–Sato transform:

$$\begin{array}{ccc} D^{\mathrm{b}}_{G(\mathbf{w})}(E(\mathbf{w})) & \xrightarrow{\mathbb{T}} & D^{\mathrm{b}}_{G(\mathbf{w}^*)}(E(\mathbf{w}^*)) \\ \mathcal{F} & \longmapsto & q_{2!}q_1^*(\mathcal{F})[\dim E(\mathbf{w})] \end{array}$$

for simple perverse sheaves $\mathcal{F}$?

Outline

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Quiver representations

Consider the type A_n equioriented quiver

$$Q_n = \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet.$$

A **quiver representation** is:

- A finite-dimensional $\mathbb{C}$-vector space M_i for each vertex.
- A linear map x_i for each arrow.

$$M_1 \xrightarrow{x_1} M_2 \xrightarrow{x_2} \cdots \xrightarrow{x_{n-1}} M_n$$

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$\mathrm{Rep}(Q_n)$ - abelian category of finite-dimensional complex representations of Q_n

Above, $\underline{\dim}(M) = (\dim M_1, \dim M_2, \ldots, \dim M_n) \in \mathbb{Z}^n_{\geqslant 0}$.

Quiver representation varieties

Fix a dimension vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)$.

A **quiver representation variety** $E(\mathbf{w})$ is the space of all quiver representations for a fixed dimension vector $\mathbf{w}$.

Note that $E(\mathbf{w})$ is an affine variety:

$$E(\mathbf{w}) \simeq \mathbb{A}^{w_1 w_2 + w_2 w_3 + \cdots + w_{n-1} w_n}.$$

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$G(\mathbf{w}) = \mathbf{GL}(w_1) \times \cdots \times \mathbf{GL}(w_n)$ acts on $E(\mathbf{w})$ by

$$(g_1, \ldots, g_n) \cdot (x_1, \ldots, x_{n-1}) = (g_2 x_1 g_1^{-1}, \ldots, g_n x_{n-1} g_{n-1}^{-1})$$

giving it a stratification by orbits.

Note that two points $x, y \in E(\mathbf{w})$ are in the same $G(\mathbf{w})$-orbit if and only if they are isomorphic objects of $\mathrm{Rep}(Q_n)$.

Classifying the orbits

Theorem (Gabriel's Theorem)

There is a bijection

$$\{\textit{indec. objects in } \mathrm{Rep}(Q_n)\}/\sim \ \xleftrightarrow{1-1}\ \{\textit{pos. roots for } A_n \textit{ root system}\}.$$

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To an indecomposable representation

$$R_{ij} = 0 \to \cdots \to 0 \to \underset{\text{vertex } i}{\mathbb{C}} \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} \underset{\text{vertex } j}{\mathbb{C}} \to 0 \to \cdots \to 0.$$

we associate its dimension vector, the positive root

$$\gamma_{ij} = (0, \ldots, 0, \underset{\text{position } i}{1}, \ldots, \underset{\text{position } j}{1}, 0, \ldots, 0).$$

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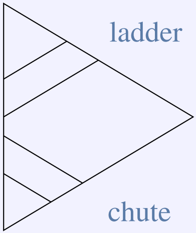
Corollary

There is a bijection

$$\{G(\mathbf{w})\textit{-orbits in } E(\mathbf{w})\} \xleftrightarrow{1-1} B(\mathbf{w}) := \{b_{ij} \mid \sum b_{ij}\gamma_{ij} = \mathbf{w}\}.$$

Outline

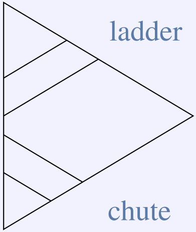
Triangular arrays



Define the set $\mathbf{P}(\mathbf{w})$ of triangular arrays of nonnegative integers such that:

- $\forall j$, the entries in the j^{th} chute sum to w_j.
- Ladders are weakly decreasing.

Triangular arrays



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We will write y_{ij} for the entry in the i^{th} chute and j^{th} column.

There is a partial order on $\mathbf{P}(\mathbf{w})$ defined by

$$Y \leqslant_{\text{comb}} Y' \iff \text{for all } i \text{ and } j, \sum_{k=1}^{j} y_{ij} \leqslant \sum_{k=1}^{j} y'_{ij}.$$

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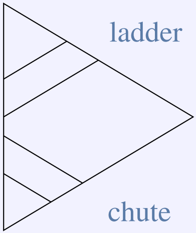

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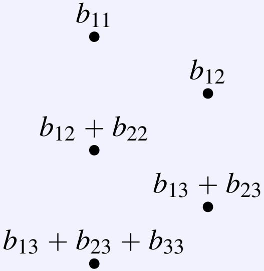
If $Y \in \mathbf{P}(\mathbf{w})$ and $Z \in \mathbf{P}(\mathbf{v})$, then we can form the entry-wise sum $Y + Z \in \mathbf{P}(\mathbf{w} + \mathbf{v})$.

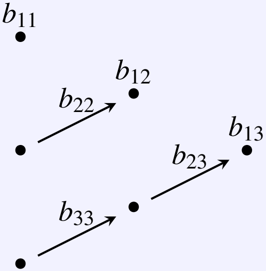
Classifying the orbits combinatorially

Lemma (Achar–Kulkarni–M.)

There is a bijection

$$B(\mathbf{w}) := \{b_{ij} \mid \sum b_{ij}\gamma_{ij} = \mathbf{w}\} \xleftrightarrow{1-1} \mathbf{P}(\mathbf{w}).$$

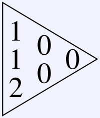




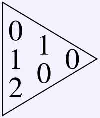
Running Example (A_3)

Let $\mathbf{w} = (1, 1, 2)$.

$$\mathbb{C} \xrightarrow{\;0\;} \mathbb{C} \xrightarrow{\;0\;} \mathbb{C}^2$$



$$\mathbb{C} \xrightarrow{\text{rank } 1} \mathbb{C} \xrightarrow{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} \mathbb{C}^2$$



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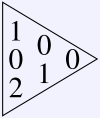

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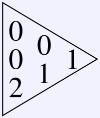

Some observations from the combinatorics

If $Y \in \mathbf{P}(\mathbf{w})$, we write $\mathcal{O}_Y$ for the corresponding $G(\mathbf{w})$-orbit in $E(\mathbf{w})$.

Denote by $M(Y)$ a representation in the orbit $\mathcal{O}_Y$.

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Lemma (Achar–Kulkarni–M.)

1. *$\mathcal{O}_Y$ is the unique closed orbit in $E(\mathbf{w})$ if and only if Y is the unique minimal element of $\mathbf{P}(\mathbf{w})$.*
2. *If $Y \in \mathbf{P}(\mathbf{w})$ and $Z \in \mathbf{P}(\mathbf{v})$, then $M(Y+Z) \simeq M(Y) \oplus M(Z)$.*
3. *$M(Y)$ is an injective object in $\mathrm{Rep}(Q_n)$ if and only if Y is constant along ladders.*
4. *$M(Y)$ is a projective object in $\mathrm{Rep}(Q_n)$ if and only if Y has nonzero entries only in the last ladder.*

Outline

Fourier–Sato transform

See Kashiwara–Schapira (Section 3.7) for more details.

Can we give a combinatorial description of the Fourier–Sato transform:

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$$\begin{array}{ccccc} & & E(\mathbf{w}) \times E(\mathbf{w}^*) & & \\ & & \uparrow & & \\ & & \{(x,y) \in E(\mathbf{w}) \times E(\mathbf{w}^*) \mid \mathrm{Re}(\langle x, y\rangle) \leqslant 0\} & & \\ & \swarrow^{q_1} & & \searrow^{q_2} & \\ E(\mathbf{w}) & & & & E(\mathbf{w}^*) \end{array}$$

Some properties and applications of the Fourier transform

Properties:

- t-exact for the perverse t-structure and sends simples to simples.
- equivalence of categories
- "almost" an involution
- compatible with convolution; i.e. $\mathbb{T}(\mathcal{F} \star \mathcal{G}) = \mathbb{T}(\mathcal{F}) \star \mathbb{T}(\mathcal{G})$.

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Applications:

- Used in the 1980s to shorten Deligne’s proof of the Weil conjectures (Laumon).
- the Springer correspondence (Hotta–Kashiwara, Evens–Mirković)
- character sheaves (Lusztig, Mirković)
- character formula for quantum loop algebras uses Fourier transform on graded quiver varieties (Nakajima)

Outline

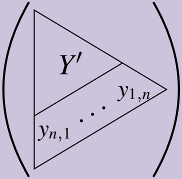
Combinatorial Fourier transform

Theorem (Achar–Kulkarni–M.)

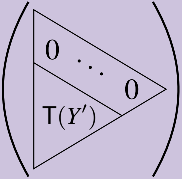
There is a bijection

$$\mathbf{P}(\mathbf{w}) \xrightarrow{\mathsf{T}} \mathbf{P}(\mathbf{w}^*)$$

defined inductively by

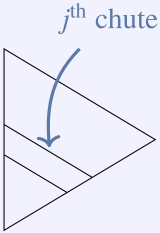
T 

$$= \tau_n^{y_{1,n}} \tau_{n-1}^{y_{2,n-1}-y_{1,n}} \dots \tau_1^{y_{n,1}-y_{n-1,2}}$$



where $\mathsf{T}(a) = a$.

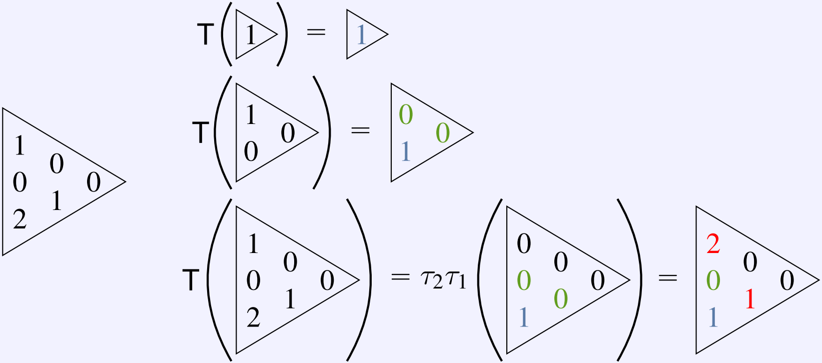
Sliding at j



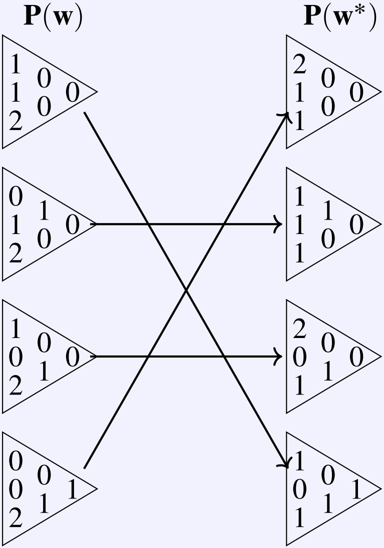
Define $\tau_j : \mathbf{P}(\mathbf{w}) \to \mathbf{P}(\mathbf{w} + \mathbf{e}_1 + \ldots + \mathbf{e}_j)$ by:

- Add 1 as far down the j^{th} chute as possible, drawing an impassable vertical line there.
- Repeat for chutes $j - 1, \ldots, 1$ not crossing lines.

Example of T



Running example



Main conjecture

Conjecture (Achar–Kulkarni–M.)

The bijection $\mathsf{T} : \mathbf{P}(\mathbf{w}) \to \mathbf{P}(\mathbf{w}^*)$ *determines* $\mathbb{T} : D^{\mathsf{b}}_{G(\mathbf{w})}(E(\mathbf{w})) \to D^{\mathsf{b}}_{G(\mathbf{w}^*)}(E(\mathbf{w}^*))$ *for simple perverse sheaves; that is,*

$$\mathbb{T}(\mathrm{IC}(\mathcal{O}_Y)) = \mathrm{IC}(\mathcal{O}_{\mathsf{T}(Y)}).$$

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Proof idea:

Since $\mathbb{T}(\mathcal{F} \star \mathcal{G}) = \mathbb{T}(\mathcal{F}) \star \mathbb{T}(\mathcal{G})$, we can use induction on the dimension vector. The proof should follow from a careful study of the combinatorics of $\star$ as well as the interplay between $\leqslant_{\mathrm{geom}}$ and $\leqslant_{\mathrm{comb}}$.

The End

Thanks!