COMPUTING UPPER CLUSTER ALGEBRAS

JACOB MATHERNE AND GREG MULLER

Abstract. This paper develops techniques for producing presentations of upper cluster algebras. These techniques are suited to computer implementation, and will always succeed when the upper cluster algebra is totally coprime and finitely generated. We include several examples of presentations produced by these methods.

1. Introduction

1.1. **Cluster algebras.** Many notable varieties have a *cluster structure*, in the following sense. They are equipped with distinguished regular functions called *cluster variables*, which are grouped into *clusters*, each of which form a transcendence basis for the field of rational functions. Each cluster is endowed with *mutation* rules for moving to other clusters, and in this way, every cluster can be reconstructed from any other cluster. Example where this occurs include semisimple Lie groups [BFZ05], Grassmannians [Sco06], partial flag varieties [GLS08], moduli of certain local systems [FG06], and others.

Given a some cluster structure, the obvious algebra to consider is the *cluster algebra* $\mathcal{A}$, the ring generated by the cluster variables.[1] However, from a geometric perspective, the more natural algebra to consider is the *upper cluster algebra* $\mathcal{U}$, defined by intersecting certain Laurent rings (see Remark 3.2.2 for the explicit geometric interpretation).

The *Laurent phenomenon* guarantees that $\mathcal{A} \subseteq \mathcal{U}$. This can be strengthened to an equality $\mathcal{A} = \mathcal{U}$ in many of the geometric examples and simpler classes of cluster algebras (such as acyclic and locally acyclic cluster algebras [BFZ05, Mul13]). In most cases where $\mathcal{A} = \mathcal{U}$ is known, the structures and properties of the algebra $\mathcal{A} = \mathcal{U}$ are fairly well-understood; for example, [BFZ05, Corollary 1.21] presents an acyclic cluster algebra as a finitely generated complete intersection.

However, there are examples where $\mathcal{A} \subsetneq \mathcal{U}$; the standard counterexample is the *Markov cluster algebra* (see Remark 6.2.1 for details). In these examples, both $\mathcal{A}$ and $\mathcal{U}$ are more difficult to work with directly, and either can exhibit pathologies. For example, the Markov cluster algebra is non-Noetherian [Mul13], and Speyer recently produced a non-Noetherian upper cluster algebra [Spe13].

1.2. **Presenting upper cluster algebras.** Nevertheless, because of its geometric nature, the authors expect that an upper cluster algebra $\mathcal{U}$ is generally better behaved than its cluster algebra $\mathcal{A}$. This is supported in the few concretely understood examples where $\mathcal{A} \subsetneq \mathcal{U}$; however, the scarcity of examples makes investigating $\mathcal{U}$ difficult.

2010 *Mathematics Subject Classification.* Primary 13F60, Secondary 14Q99.

Keywords: Cluster algebras, upper cluster algebras, presentations of algebras, computational algebra.

The second author was supported by the VIGRE program at LSU, National Science Foundation grant DMS-0739382.

[1]Technically, the construction of the cluster algebra used in this note includes the inverses to a finite set.

The goal of this note is to alleviate this problem by developing techniques to produce explicit presentations of $\mathcal{U}$. The main tool is the following lemma, which gives several computationally distinct criteria for when a Noetherian ring $\mathcal{S}$ is equal to $\mathcal{U}$.

Lemma 1.2.1. *If $\mathcal{A}$ is a cluster algebra with deep ideal $\mathbb{D}$, and $\mathcal{S}$ is a Noetherian ring such that $\mathcal{A} \subseteq \mathcal{S} \subseteq \mathcal{U}$, then the following are equivalent.*

(1) $\mathcal{S} = \mathcal{U}$.
(2) $\mathcal{S}$ is normal and codim$(\mathcal{S}\mathbb{D}) \geq 2$.
(3) $\mathcal{S}$ is $S2$ and codim$(\mathcal{S}\mathbb{D}) \geq 2$.
(4) $Ext^1_{\mathcal{S}}(\mathcal{S}/\mathcal{S}\mathbb{D}, \mathcal{S}) = 0$.
(5) $\mathcal{S}f = (\mathcal{S}f : (\mathcal{S}\mathbb{D})^\infty)$ for every product $f = x_1x_2...x_m$ of the variables in a cluster.
(6) $\mathcal{S}f = (\mathcal{S}f : (\mathcal{S}\mathbb{D})^\infty)$ for any product $f = x_1x_2...x_m$ of the variables in a cluster.

If $\mathcal{S}f \neq (\mathcal{S}f : (\mathcal{S}\mathbb{D})^\infty)$, then $(\mathcal{S}f : (\mathcal{S}\mathbb{D})^\infty)f^{-1}$ contains elements of $\mathcal{U}$ not in $\mathcal{S}$.

Here, the *deep ideal* $\mathbb{D}$ is the ideal in $\mathcal{A}$ generated by the products of the mutable cluster variables in each cluster (see Section 3.2).

This lemma is constructive, in that a negative answer to condition (5) explicitly provides new elements of $\mathcal{U}$. Even without a clever guess for a generating set of $\mathcal{U}$, iteratively checking this criterion and adding new elements can produce a presentation for $\mathcal{U}$. Speyer's example demonstrates that this algorithm cannot always work; however, if $\mathcal{U}$ is finitely generated, this approach will always produce a generating set (Corollary 5.2.2).

Naturally, we include several examples of these explicit presentations. Sections 6 and 7 contain presentations for the upper cluster algebras of the seeds pictured in Figure 1.

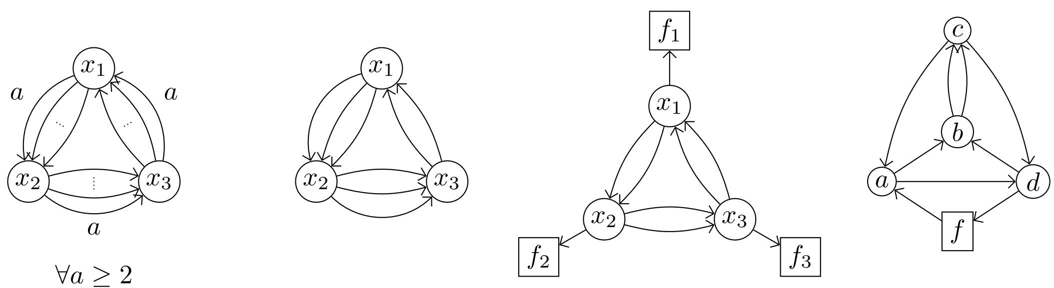


FIGURE 1. Seeds of upper cluster algebras presented in this note.

Remark 1.2.2. To compute examples, we use a variation of Lemma 4.4.1 involving *lower bounds* and *upper bounds*, which requires that our cluster algebras are *totally coprime*.

2. Cluster algebras

Cluster algebras are a class of commutative unital domains. Up to a finite localization, they are generated in their field of fractions by distinguished elements, called *cluster variables*. The cluster variables (and hence the cluster algebra) are produced by an recursive procedure, called *mutation*. While cluster algebras are geometrically motivated, their construction is combinatorial and determined by some simple data called a 'seed'.

2.1. **Ordered seeds.** A matrix $\mathsf{M} \in \mathrm{Mat}_{m,m}(\mathbb{Z})$ is *skew-symmetrizable* if there is a non-negative, diagonal matrix $\mathsf{D} \in \mathrm{Mat}_{m,m}(\mathbb{Z})$ such that DM is skew-symmetric; that is, that $(\mathsf{DM})^\top = -\mathsf{DM}$.

Let $n \geq m \geq 0$ be integers, and let $\mathsf{B} \in \mathrm{Mat}_{n,m}(\mathbb{Z})$ be an integer valued $n \times m$-matrix. Let $\mathsf{B}^0 \in \mathrm{Mat}_{m,m}(\mathbb{Z})$ be the *principal part*, the submatrix of B obtained by deleting the last $n - m$ rows.

An **ordered seed** is a pair $(\mathbf{x}, \mathsf{B})$ such that...

- $\mathsf{B} \in \mathrm{Mat}_{n,m}(\mathbb{Z})$,
- B^0 is skew-symmetrizable, and
- $\mathbf{x} = (x_1, ..., x_n)$ is an n-tuple of elements in a field $\mathcal{F}$ of characteristic zero, which is a free generating set for $\mathcal{F}$ as a field over $\mathbb{Q}$.

The various parts of an ordered seed have their own names.

- The matrix B is the *exchange matrix*.
- The n-tuple $\mathbf{x}$ is the *cluster*.
- Elements $x_i \in \mathbf{x}$ are *cluster variables*.[2] These are further subdivided by index.
  - If $0 < i \leq m$, x_j is a *mutable variable*.
  - If $m < i \leq n$, x_j is a *frozen variable*.

The ordering of the cluster variables in $\mathbf{x}$ is a matter of convenience. A permutation of the cluster variables which preserves the flavor of the cluster variable (mutable/frozen) acts on the ordered seed by reordering $\mathbf{x}$ and conjugating B.

A skew-symmetric seed $(\mathbf{x}, \mathsf{B})$ can be diagrammatically encoded as an *ice quiver* (Figure 2). Put each mutable variable x_i in a circle, and put each frozen variable x_i in a square. For each pair of indices $i < j$ with $i \leq m$, add B_{ji} arrows from i to j, where 'negative arrows' go from j to i.

$$\left(\mathbf{x} = \{x_1, x_2, x_3\}, \mathsf{B} = \begin{bmatrix} 0 & -3 \\ 3 & 0 \\ -2 & 1 \end{bmatrix} \right) \qquad \mathsf{Q} =$$

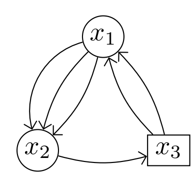


FIGURE 2. The ice quiver associated to a seed

A seed $(\mathbf{x}, \mathsf{B})$ is called *acyclic* if Q does not contain a directed cycle of mutable vertices. The seed in Figure 2 is acyclic.

2.2. **Cluster algebras.** Given an ordered seed $(\mathbf{x}, \mathsf{B})$ and some $1 \leq k \leq m$, define the *mutation* of $(\mathbf{x}, \mathsf{B})$ at k to be the ordered seed $(\mathbf{x}', \mathsf{B}')$, where

$$x_i' := \left\{ \begin{array}{cc} \left(\prod_{\mathsf{B}_{jk}>0} x_j^{\mathsf{B}_{jk}} + \prod_{\mathsf{B}_{jk}<0} x_j^{-\mathsf{B}_{jk}}\right) x_i^{-1} & \text{if } i = k \\ x_i & \text{otherwise} \end{array} \right\}$$

$$\mathsf{B}_{ij}' := \left\{ \begin{array}{cc} -\mathsf{B}_{ij} & \text{if } i = k \text{ or } j = k \\ \mathsf{B}_{ij} + \frac{|\mathsf{B}_{ik}|\mathsf{B}_{kj} + \mathsf{B}_{ik}|\mathsf{B}_{kj}|}{2} & \text{otherwise} \end{array} \right\}$$

Since $(\mathbf{x}', \mathsf{B}')$ is again an ordered seed, mutation may be iterated at any sequence of indices in $1, 2, ..., m$. Mutation twice in a row at k returns to the original ordered seed.

---

[2]Many authors do not consider frozen variables to be cluster variables, instead referring to them as 'geometric coefficients', following [FZ07].

Two ordered seeds $(\mathbf{x}, \mathsf{B})$ and $(\mathbf{y}, \mathsf{C})$ are *mutation-equivalent* if $(\mathbf{y}, \mathsf{C})$ is a obtained from $(\mathbf{x}, \mathsf{B})$ by a sequence of mutations and permutations.

Definition 2.2.1. Given an ordered seed $(\mathbf{x}, \mathsf{B})$, the associated **cluster algebra** $\mathcal{A}(\mathbf{x}, \mathsf{B})$ is the subring of the ambient field $\mathcal{F}$ generated by

$$\{x_i^{-1} \mid m < i \leq n\} \cup \bigcup_{(\mathbf{y}, \mathsf{C}) \sim (\mathbf{x}, \mathsf{B})} \mathbf{y}$$

A *cluster variable* in $\mathcal{A}(\mathbf{x}, \mathsf{B})$ is a cluster variable in any ordered seed mutation-equivalent to $(\mathbf{x}, \mathsf{B})$, and it is mutable or frozen based on its index in any seed. A *cluster* in $\mathcal{A}(\mathbf{x}, \mathsf{B})$ is a set of cluster variables appearing as the cluster in some ordered seed. Mutation-equivalent seeds define the same cluster algebra $\mathcal{A}$. The seed will often be omitted from the notation when clear.

A cluster algebra $\mathcal{A}$ is **acyclic** if there exists an acyclic seed of $\mathcal{A}$; usually, an acyclic cluster algebra will have many non-acyclic seeds as well. Acyclic cluster algebras have proven to be the most easily studied class; for example, [BFZ05, Corollary 1.21] gives a presentation of $\mathcal{A}$ with $2n$ generators and n relations.

2.3. **Upper cluster algebras.** A basic tool in the theory of cluster algebras is the following theorem, usually called the *Laurent phenomenon.*

Theorem 2.3.1 (Theorem 3.1, [FZ02])**.** *Let $\mathcal{A}$ be a cluster algebra, and $\mathbf{x} = \{x_1, x_2, ..., x_n\}$ be a cluster in $\mathcal{A}$. As subrings of $\mathcal{F}$,*

$$\mathcal{A} \subset \mathbb{Z}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$$

This is the localization of $\mathcal{A}$ at the mutable variables $x_1, ..., x_m$.

The theorem says elements of $\mathcal{A}$ can be expressed as Laurent polynomials in many different sets of variables (one such expression for each cluster). The set of all rational functions in $\mathcal{F}$ with this property is an important algebra in its own right, and the central object of study in this note.

Definition 2.3.2. Given a cluster algebra $\mathcal{A}$, the **upper cluster algebra** $\mathcal{U}$ is defined

$$\mathcal{U} := \bigcap_{\substack{\text{clusters} \\ \mathbf{x} = \{x_1, ..., x_n\} \text{ in } \mathcal{A}}} \mathbb{Z}[x_1^{\pm 1}, ..., x_n^{\pm 1}] \subset \mathcal{F}$$

The Laurent phenomenon is equivalent to the containment $\mathcal{A} \subseteq \mathcal{U}$.

Proposition 2.3.3. [Mul13, Proposition 2.1] *Upper cluster algebras are normal.*

Remark 2.3.4. Any intersection of normal domains in their fraction field is normal.

2.4. **Lower and upper bounds.** The cluster algebras that have finitely many clusters have an elegant classification by Dynkin diagrams [FZ03]. However, such *finite-type* cluster algebras are quite rare; even the motivating examples are frequently *infinite-type.* Working with infinite-type $\mathcal{A}$ or $\mathcal{U}$ can be daunting because the definitions involve infinite generating sets or intersections (this is especially a problem for computer computations).

Following [BFZ05], to any seed $\mathbf{x}$, we associate bounded analogs of $\mathcal{A}$ and $\mathcal{U}$ called *lower* and *upper bounds.* The definitions are the same, except the only seeds considered are $\mathbf{x}$ and those seeds a single mutation away from $\mathbf{x}$.

As a standard abuse of notation, for a fixed seed $(\mathbf{x} = \{x_1, x_2, ..., x_n\}, \mathsf{B})$, let x_i' denote the mutation of x_i in $(\mathbf{x}, \mathsf{B})$.

Definition 2.4.1. Let $(\mathbf{x}, \mathsf{B})$ be a seed in $\mathcal{F}$.

The **lower bound** $L_{\mathbf{x}}$ is the subring of $\mathcal{F}$ generated by $\{x_1, x_2, ..., x_n\}$, the one-step mutations $\{x_1', x_2', ..., x_m'\}$, and the inverses to invertible frozen variables $\{x_{m+1}^{-1}, ..., x_n^{-1}\}$.

The **upper bound** $\mathcal{U}_{\mathbf{x}}$ is the intersection in $\mathcal{F}$ of the $n+1$ Laurent rings corresponding to $\mathbf{x}$ and its one-step mutations.

$$\mathcal{U}_{\mathbf{x}} := \mathbb{Z}[x_1^{\pm 1}, ..., x_n^{\pm 1}] \cap \bigcap_i \mathbb{Z}[x_1^{\pm 1}, ..., x_{i-1}^{\pm 1}, x_i'^{\pm 1}, x_{i+1}^{\pm 1} ..., x_n^{\pm 1}]$$

The names ‘lower bound’ and ‘upper bound’ are justified by the obvious inclusions

$$L_{\mathbf{x}} \subseteq \mathcal{A} \subseteq \mathcal{U} \subseteq \mathcal{U}_{\mathbf{x}}$$

When does $\mathcal{U} = \mathcal{U}_{\mathbf{x}}$? A seed $(\mathbf{x}, \mathsf{B})$ is *coprime* if every pair of columns in B are linearly independent. A cluster algebra is **totally coprime** if every seed is coprime.

Theorem 2.4.2 (Corollary 1.7, [BFZ05]). *If $\mathcal{A}$ is totally coprime, then $\mathcal{U} = \mathcal{U}_{\mathbf{x}}$ for any seed $(\mathbf{x}, \mathsf{B})$.*

Mutating a seed can make coprime seeds non-coprime (and vice versa), so verifying a cluster algebra is totally coprime may be hard in general. A stronger condition is that the exchange matrix B has *full rank* (ie, kernel 0); this is preserved by mutation, so it implies the cluster algebra $\mathcal{A}(\mathsf{B})$ is totally coprime.

Theorem 2.4.3 (Proposition 1.8, [BFZ05]). *If the exchange matrix B of a seed of $\mathcal{A}$ is full rank, then $\mathcal{A}$ is totally coprime.*

Of course, there are many totally coprime cluster algebras which are not full rank.[3]

3. Regular functions on an open subscheme

This section collects some generalities about the ring we denote $\Gamma(R, I)$ – the ring of regular functions on the open subscheme of *Spec*(R) whose complement is $V(I)$ – and relates this idea to cluster algebras. Throughout this section, let R be a domain with fraction field $\mathcal{F}(R)$.[4]

3.1. **Definition.** For any ideal $I \subset R$, define the ring $\Gamma(R, I)$ as the intersection (taken in $\mathcal{F}(R)$)

$$\Gamma(R, I) := \bigcap_{r \in I \backslash \{0\}} R[r^{-1}]$$

Remark 3.1.1. In geometric terms, $\Gamma(R, I)$ is the ring of rational functions on *Spec*(R) which are regular on the complement of $V(I)$. As a consequence, $\Gamma(R, I)$ only depends on I up to radical. Neither of these facts are necessary for the rest of this note, however.

Proposition 3.1.2. *If I is generated by a set $\pi \subset R \backslash \{0\}$, then*

$$\Gamma(R, I) = \bigcap_{r \in \pi} R[r^{-1}]$$

[3] Proposition 6.1.2 provides a class of such examples.

[4] All rings in this note are commutative and unital, but need not be Noetherian.

Proof. Choose some $f \in I$, and write $f = \sum_{r\in\pi_0} b_r r$, where π_0 is a finite subset of π. Let $g \in \bigcap_{r\in\pi} R[r^{-1}]$; therefore, there are $n_r \in R$ and $\alpha_r \in \mathbb{N}$ such that $g = \frac{n_r}{r^{\alpha_r}}$ for all $r \in \pi$. Define

$$\beta = 1 + \sum_{r\in\pi_0} \alpha_r$$

and consider $f^\beta g$. Expanding $f^\beta = \left(\sum b_r r\right)^\beta$, every monomial expression in the $\{r\}$ contains at least one $r' \in \pi_0$ with exponent greater or equal to $\alpha_{r'}$. Since ${r'}^{\alpha_{r'}} g = n_{r'} \in R$, it follows that $f^\beta g \in R$ and $g \in R[f^{-1}]$. Therefore, $\bigcap_{r\in\pi} R[r^{-1}] \subseteq \Gamma(R, I)$. □

Proposition 3.1.3. *If* $R \subseteq S \subseteq \Gamma(R, I)$*, then* $\Gamma(R, I) = \Gamma(S, SI)$*.*

Proof. For $i \in I$, $\Gamma(R, I) \subset R[r^{-1}]$, and so $S \subset R[r^{-1}]$. Then $S[r^{-1}] = R[r^{-1}]$ for all $r \in I$. If π generates I over R, then π generates SI over S. By Proposition 3.1.2,

$$\Gamma(R, I) = \bigcap_{r\in\pi} R[r^{-1}] = \bigcap_{r\in\pi} S[r^{-1}] = \Gamma(S, SI)$$

This completes the proof. □

3.2. **Upper cluster algebras.** The relation between a cluster algebra $\mathcal{A}$ and its upper cluster algebra $\mathcal{U}$ is an example of this construction. Define the **deep ideal** $\mathbb{D}$ of $\mathcal{A}$ by

$$\mathbb{D} := \sum_{\text{clusters } \{x_1,x_2,\ldots,x_n\}} \mathcal{A} x_1 x_2 \ldots x_m$$

That is, it is the $\mathcal{A}$-ideal generated by the product of the mutables variables in each cluster.

Proposition 3.2.1. $\Gamma(\mathcal{A}, \mathbb{D}) = \mathcal{U}$*.*

Proof. Since $\mathbb{D}$ is generated by the products of the mutable variables in the clusters,

$$\begin{aligned}
\Gamma(\mathcal{A}, \mathbb{D}) &= \bigcap_{\text{clusters } \{x_1,x_2,\ldots,x_n\}} \mathcal{A}[(x_1 x_2 \ldots x_m)^{-1}] \\
&= \bigcap_{\text{clusters } \{x_1,x_2,\ldots,x_n\}} \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]
\end{aligned}$$

Thus, $\Gamma(\mathcal{A}, \mathbb{D}) = \mathcal{U}$. □

Remark 3.2.2. The proposition is equivalent to the following well-known geometric interpretation of $\mathcal{U}$. If $\{x_1, \ldots, x_n\}$ is a cluster, then the isomorphism

$$\mathcal{A}[(x_1 x_2 \ldots x_m)^{-1}] \simeq \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$$

determines an open inclusion $\mathbb{G}_{\mathbb{Z}}^n \hookrightarrow Spec(\mathcal{A})$.[5] The union of all such open affine subschemes is a smooth open subscheme in $Spec(\mathcal{A})$, whose complement is $V(\mathbb{D})$.[6] The proposition states that $\mathcal{U}$ is the ring of regular functions on this union.

[5]These open algebraic tori are called *toric charts* in [Sco06] and *cluster tori* in [Mul13].

[6]This union is called the *cluster manifold* in [GSV03].

3.3. **Upper bounds.** Let $(\mathbf{x},\mathsf{B})$ be a seed with $\mathbf{x}=\{x_1,x_2,...,x_n\}$. As in Section 2.4, let x_i' denote the mutation of x_i in $\mathbf{x}$. The **lower deep ideal** $\mathbb{D}_{\mathbf{x}}$ is the $L_{\mathbf{x}}$-ideal

$$\mathbb{D}_{\mathbf{x}} := L_{\mathbf{x}}(x_1x_2...x_m) + \sum_i L_{\mathbf{x}}(x_1x_2...x_{i-1}x_i'x_{i+1}...x_m)$$

Proposition 3.2.1 has an analog.

Proposition 3.3.1. $\Gamma(L_{\mathbf{x}},\mathbb{D}_{\mathbf{x}})=\mathcal{U}_{\mathbf{x}}$

Proof. Since $\mathbb{D}_{\mathbf{x}}$ is generated by the products of the mutable variables in $L_{\mathbf{x}}$,

$$\begin{aligned}\Gamma(L_{\mathbf{x}},\mathbb{D}_{\mathbf{x}}) &= L_{\mathbf{x}}[(x_1x_2...x_m)^{-1}]\cap\bigcap_i L_{\mathbf{x}}[(x_1x_2...x_i'...x_m)^{-1}]\\ &= \mathbb{Z}[x_1^{\pm1},...,x_n^{\pm1}]\cap\bigcap_i \mathbb{Z}[x_1^{\pm1},...,x_i'^{\pm1},...,x_n^{\pm1}]\end{aligned}$$

Thus, $\Gamma(L_{\mathbf{x}},\mathbb{D}_{\mathbf{x}})=\mathcal{U}_{\mathbf{x}}$. □

In practice, $\Gamma(L_{\mathbf{x}},\mathbb{D}_{\mathbf{x}})$ is much easier to work with than $\Gamma(\mathcal{A},\mathbb{D})$, because the objects involved are defined by finite generating sets.

Remark 3.3.2. For any set of clusters S in $\mathcal{A}$, one may define L_S generated by the variables in S, $\mathcal{U}_S$ as the intersection of the Laurent rings of clusters in S, and $\mathbb{D}_S$ an ideal in L_S generated by the products of clusters in S. Again, one has $\mathcal{U}_S=\Gamma(L_S,\mathbb{D}_S)$.

4. Criteria for $\Gamma(R,I)$

Given a 'guess' for $\Gamma(R,I)$ – a domain S such that $R\subseteq S\subseteq\Gamma(R,I)$ – there are several criteria for verifying if $S=\Gamma(R,I)$. This section develops these criteria.

4.1. **Saturations.** Given two ideals I,J in a domain R, define the **saturation**

$$(J:I^\infty)=\{r\in R\mid \forall g\in I,\ \exists n\in\mathbb{N}\text{ s.t. } rg^n\in J\}$$

Computer algebra programs can compute saturations when R is finitely generated.

Remark 4.1.1. When I is not finitely generated, this definition of saturation may differ from the infinite union $\bigcup_n(J:I^n)$, which amounts to reversing the order of quantifiers.

Saturations can be used to compute the sub-R-module of $\Gamma(R,I)$ with denominator f.

Proposition 4.1.2. *If* $f\in R\backslash\{0\}$*, then*

$$Rf^{-1}\cap\Gamma(R,I)=(Rf:I^\infty)f^{-1}$$

Proof. If $g\in R\cap f\Gamma(R,I)$, then for any $r\in I\backslash\{0\}$, we may write $gf^{-1}=hr^{-m}$ for some $h\in R$ and $m\in\mathbb{N}$. Then $gr^m=hf\in Rf$; and so $g\in(Rf:I^\infty)$.

If $g\in(Rf:I^\infty)$, then for any $r\in I$, there is some m such that $gr^m\in Rf$. It follows that $gf^{-1}\in Rr^{-m}\subset R[r^{-1}]$. Therefore, $gf^{-1}\in\Gamma(R,I)$, and so $g\in f\Gamma(R,I)$. □

Saturations can also detect when $R=\Gamma(R,I)$.

Proposition 4.1.3. *Let* $f\in I\backslash\{0\}$*. Then* $R=\Gamma(R,I)$ *if and only if* $Rf=(Rf:I^\infty)$*.*

Proof. If $R=\Gamma(R,I)$, then $(Rf:I^\infty)=Rf\cap\Gamma(R,I)=Rf$.

Assume $Rf=(Rf:I^\infty)$. Let $g\in\Gamma(R,I)$, and let n be the smallest integer such that $f^ng\in R$. If $n\geq1$, then

$$f(f^{n-1}g)\in R\cap(f\Gamma(R,I))=(Rf:I^\infty)=Rf$$

and so $f^{n-1}g\in R$, contradicting minimality of n. So $g\in R$, and so $\Gamma(R,I)=R$. □

4.2. **The saturation criterion.** Given a ring S with $R \subseteq S \subseteq \Gamma(R,I)$, the following lemma gives a necessary and computable criterion for when $S = \Gamma(R,I)$. Perhaps more importantly, if $S \subsetneq \Gamma(R,I)$, it explicitly gives new elements of $\Gamma(R,I)$, which can be used to generate a better guess $S' \subseteq \Gamma(R,I)$.

Lemma 4.2.1. *Let $R \subseteq S \subseteq \Gamma(R,I)$. For any $f \in I \backslash \{0\}$,*

$$S \subseteq (Sf : (SI)^{\infty})f^{-1} \subset \Gamma(R,I)$$

Furthermore, either

- *$S = \Gamma(R,I)$, or*
- *$S \subsetneq (Sf : (SI)^{\infty})f^{-1} \subseteq \Gamma(R,I)$.*

Proof. By Proposition 3.1.3, $\Gamma(R,I) = \Gamma(S,SI)$. The containment $(Sf : (SI)^{\infty})f^{-1} \subset \Gamma(R,I)$ follows from Proposition 4.1.2. The containment $S \subseteq (Sf : (SI)^{\infty})f^{-1}$ is clear from the definition of the saturation. If $(Sf : (SI)^{\infty})f^{-1} = S$, then Proposition 4.1.3 implies that $S = \Gamma(R,I)$. □

4.3. **Noetherian algebraic criteria.** When the ring S is Noetherian, there are several alternative criteria to verify that $S = \Gamma(R,I)$.[7] When S is also normal, these criteria are sharp, but none of them can give a constructive negative answer similar to Lemma 4.2.1.

The definitions of 'codimension', 'S2' and 'depth' used here are found in [Eis95].

Lemma 4.3.1. *Let $R \subseteq S \subseteq \Gamma(R,I)$. If S is Noetherian, then each of the following statements implies the next.*

(1) S is normal and codim$(SI) \geq 2$.
(2) S is S2 and codim$(SI) \geq 2$.
(3) depth$_S(SI) \geq 2$; that is, $Ext^1_S(S/SI, S) = 0$.
(4) $S = \Gamma(R,I)$.

If S is normal and Noetherian, then the above statements are equivalent.

Proof. (1) $\Rightarrow$ (2). By Serre's criterion [Eis95, Theorem 11.5.i], a normal Noetherian domain is S2.

(2) $\Rightarrow$ (3). The S2 condition implies that every ideal of codimension ≥ 2 has depth ≥ 2; see the proof of [Eis95, Theorem 18.15].[8]

(Not 4) $\Rightarrow$ (Not 3). Assume that $S \subsetneq \Gamma(R,I)$, and let $f \in I$. By Lemma 4.2.1 and Proposition 4.1.2,

$$S \subsetneq (Sf : (SI)^{\infty})f^{-1} = Sf^{-1} \cap \Gamma(S,SI)$$

Since S is Noetherian, SI is finitely-generated, and so it is possible to find an element $g \in Sf^{-1} \cap \Gamma(S,SI)$ such that $g \notin S$ but $gI \subseteq S$. The natural short exact sequence

$$0 \to S \hookrightarrow Sg \to Sg/S \to 0$$

is an essential extension, and so $Ext^1_S(Sg/S, S) \neq 0$.

The map $S/SI \to Sg/S$ which sends 1 to g is a surjection, and its kernel K is a torsion S-module. Hence, there is a long exact sequence which contains

$$\cdots \to Hom_S(K,S) \to Ext^1_S(Sg/S,S) \to Ext^1_S(S/SI,S) \to ...$$

Since K is torsion, $Hom_S(K,S) = 0$, and so $Ext^1_S(S/SI,S) \neq 0$.

(S normal) + (Not 1) $\Rightarrow$ (Not 4). Assume that S is normal, and that codim$(SI) = 1$. Therefore, there is a prime S-ideal P containing SI with codim$(P) = 1$. By Serre's

[7] However, even when R is Noetherian, one cannot always expect that $\Gamma(R,I)$ is Noetherian.
[8] Some sources take this as the definition of S2.

criterion [Eis95, Theorem 11.5.ii], the localization S_P is a discrete valuation ring. Let $\nu : \mathcal{F}(S)^* \to \mathbb{Z}$ be the corresponding valuation.

Let $a_1, a_2, ..., a_j$ generate P over S. Then $a_1, a_2, ..., a_j$ generate $S_P P$ over S_P. There must be some a_i with $\nu(a_i) = 1$, and this element generates $S_P P$. Reindexing as needed, assume that $\nu(a_1) = 1$. For each a_i, there exists $f_i, g_i \in S - P$ such that

$$a_i = \frac{f_i}{g_i} a_1^{\nu(a_i)}$$

Let $d = gcd(\nu(a_i))$. Then, for all $1 \leq k \leq j$,

$$x := \frac{1}{a_1^d} \left(\prod_{1 < i \leq j} g_i^{\frac{d}{\nu(a_i)}} \right) = \left(\frac{f_k}{a_k^d} \right)^{\frac{d}{\nu(a_k)}} \left(\prod_{\substack{1 < i \leq j \\ i \neq k}} g_i^{\frac{d}{\nu(a_i)}} \right) \in S[a_k^{-1}]$$

It follows that $x \in \Gamma(S, P) \subseteq \Gamma(S, SI) = \Gamma(R, I)$. However, since $\nu(x) = -d$, it follows that $x \notin S$, and so $S \neq \Gamma(R, I)$. $\square$

Remark 4.3.2. The implication (1) $\Rightarrow$ (4) is one form of the ‘algebraic Hartog lemma’, in analogy with Hartog’s lemma in complex analysis.

Remark 4.3.3. The assumption that S is Noetherian is essential. If

$$R = S = \mathbb{C}[[x^t \mid t \in \mathbb{Q}_{\geq 0}]]$$

is the ring of Puiseux series without denominator, and I is generated by $\{x^t\}_{t>0}$, then R is normal and $Ext^1(R/I, R) = 0$. Nevertheless,

$$\Gamma(R, I) = \mathbb{C}[[x^t \mid t \in \mathbb{Q}]] \neq R$$

is the field of all Puiseux series.

4.4. **Criteria for $\mathcal{U}$.** We restate the previous criteria for upper cluster algebras.

Lemma 4.4.1. *If $\mathcal{A}$ is a cluster algebra with deep ideal $\mathbb{D}$, and $\mathcal{S}$ is a Noetherian ring such that $\mathcal{A} \subseteq \mathcal{S} \subseteq \mathcal{U}$, then the following are equivalent.*

(1) $\mathcal{S} = \mathcal{U}$.
(2) $\mathcal{S}$ is normal and codim$(\mathcal{S}\mathbb{D}) \geq 2$.
(3) $\mathcal{S}$ is $S2$ and codim$(\mathcal{S}\mathbb{D}) \geq 2$.
(4) $Ext^1_{\mathcal{S}}(\mathcal{S}/\mathcal{S}\mathbb{D}, \mathcal{S}) = 0$.
(5) $\mathcal{S}f = (\mathcal{S}f : (\mathcal{S}\mathbb{D})^\infty)$ for every product $f = x_1 x_2 ... x_m$ of the variables in a cluster.
(6) $\mathcal{S}f = (\mathcal{S}f : (\mathcal{S}\mathbb{D})^\infty)$ for any product $f = x_1 x_2 ... x_m$ of the variables in a cluster.

If $\mathcal{S}f \neq (\mathcal{S}f : (\mathcal{S}\mathbb{D})^\infty)$, then $(\mathcal{S}f : (\mathcal{S}\mathbb{D})^\infty) f^{-1}$ contains elements of $\mathcal{U}$ not in $\mathcal{S}$.

However, we are interested in infinite-type cluster algebras, where the containments $\mathcal{A} \subseteq \mathcal{S} \subseteq \mathcal{U}$ cannot be naively verified by hand or computer. This is where lower and upper bounds are helpful, since the analogous containments can be checked directly.

Lemma 4.4.2. *If $(\mathbf{x}, \mathsf{B})$ is seed in a totally coprime cluster algebra $\mathcal{A}$ and $\mathcal{S}$ a Noetherian ring such that $L_{\mathbf{x}} \subseteq \mathcal{S} \subseteq \mathcal{U}_{\mathbf{x}}$, then the following are equivalent.*

(1) $\mathcal{S} = \mathcal{U}_{\mathbf{x}} = \mathcal{U}$.
(2) $\mathcal{S}$ is normal and codim$(\mathcal{S}\mathbb{D}_{\mathbf{x}}) \geq 2$.
(3) $\mathcal{S}$ is $S2$ and codim$(\mathcal{S}\mathbb{D}_{\mathbf{x}}) \geq 2$.
(4) $Ext^1_{\mathcal{S}}(\mathcal{S}/\mathcal{S}\mathbb{D}_{\mathbf{x}}, \mathcal{S}) = 0$.
(5) $\mathcal{S}f = (\mathcal{S}f : (\mathcal{S}\mathbb{D})^\infty)$ for any product $f = x_1 x_2 ... x_m$ of the variables in $\mathbf{x}$.

If $\mathcal{S}f \neq (\mathcal{S}f : (\mathcal{S}\mathbb{D}_{\mathbf{x}})^\infty)$*, then* $(\mathcal{S}f : (\mathcal{S}\mathbb{D}_{\mathbf{x}})^\infty)f^{-1}$ *contains elements of* $\mathcal{U}$ *not in* $\mathcal{S}$*.*

Note that $\mathcal{U}_{\mathbf{x}}$ is normal by Remark 2.3.4, and so the strong form of Lemma 4.3.1 applies.

Remark 4.4.3. Criterion (2) was used implicitly in the proofs of [BFZ05, Theorem 2.10] and [Sco06, Proposition 7], and a form of it is stated in [FP13, Proposition 3.6].

5. Presenting $\mathcal{U}$

This section outlines the steps for checking if a set of Laurent polynomials generates a totally coprime upper cluster algebra $\mathcal{U}$ over the subring of frozen variables.

5.1. **From conjectural generators to a presentation.** Fix a seed $(\mathbf{x} = \{x_1, ..., x_m\}, \mathsf{B})$ in a totally coprime cluster algebra $\mathcal{A}$. Let

$$\mathbb{ZP} := \mathbb{Z}[x_{m+1}^{\pm 1}, x_{m+2}^{\pm 1}, ..., x_n^{\pm 1}]$$

be the *coefficient ring* – the Laurent ring generated by the frozen variables and their inverses.

Start with a finite set of Laurent polynomials in $\mathbb{Z}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$, which hopefully generates $\mathcal{U}$ over $\mathbb{ZP}$. We assume that all the initial mutable variables $x_1, ..., x_n$ are in this set. Write this set as

$$x_1, x_2, ...x_m, y_1, ..., y_p$$

where

$$y_i = \frac{N_i(x_1, ..., x_n)}{x_1^{\alpha_{1i}} x_2^{\alpha_{2i}} ... x_n^{\alpha_{ni}}} \in \mathbb{Z}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$$

for some polynomial $N_i(x_1, ..., x_n)$.

- **Compute the ideal of relations.** Let

$$\widetilde{\mathcal{S}} := \mathbb{ZP}[x_1, ..., x_m, y_1, ..., y_p]$$

be a polynomial ring over $\mathbb{ZP}$ (here, the y_is are just symbols). Define $\widetilde{I}$ to be the $\widetilde{\mathcal{S}}$-ideal generated by elements of the form

$$y_i(x_1^{\alpha_{1i}} x_2^{\alpha_{2i}} ... x_n^{\alpha_{ni}}) - N_i(x_1, ..., x_n)$$

as i runs from 1 to p. Let $I := (\widetilde{I} : \widetilde{\mathcal{S}}(x_1...x_m)^\infty)$ be the saturation of I by the principal $\widetilde{\mathcal{S}}$-ideal generated by the product of the mutable variables $x_1 x_2 ... x_m$.

Lemma 5.1.1. *The sub-*$\mathbb{ZP}$*-algebra of* $\mathbb{Z}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$ *generated by*

$$x_1, x_2, ...x_m, y_1, ..., y_p$$

is naturally isomorphic to the quotient $\mathcal{S} := \widetilde{\mathcal{S}}/I$*.*

Proof. Let the localization $\widetilde{\mathcal{S}}[(x_1 x_2 ... x_m)^{-1}]$ is the ring

$$\mathbb{Z}[x_1^{\pm 1}, ..., x_n^{\pm 1}, y_1, ..., y_p]$$

The induced ideal $\widetilde{\mathcal{S}}[(x_1 x_2 ... x_m)^{-1}]\widetilde{I}$ is generated by elements of the form

$$y_i - (x_1^{-\alpha_{1i}} x_2^{-\alpha_{2i}} ... x_n^{-\alpha_{ni}}) N_i(x_1, ..., x_n)$$

and so the quotient $\widetilde{\mathcal{S}}[(x_1 x_2 ... x_m)^{-1}]/\widetilde{\mathcal{S}}[(x_1 x_2 ... x_m)^{-1}]\widetilde{I}$ eliminates the y_is and is isomorphic to $\mathbb{Z}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$. The kernel of the composition

$$\widetilde{\mathcal{S}} \to \widetilde{\mathcal{S}}[(x_1 x_2 ... x_m)^{-1}] \to \mathbb{Z}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$$

consists of elements $r \in \widetilde{\mathcal{S}}$ such that $(x_1 x_2 ... x_m)^i r \in I$ for some i; this is the saturation I. $\square$

- **Verify that $L_{\mathbf{x}} \subseteq \mathcal{S} \subseteq \mathcal{U}_{\mathbf{x}}$.** For the first containment, it suffices to check that $x'_1, x'_2, ..., x'_m \in \mathcal{S}$, because the other generators of $L_{\mathbf{x}}$ are in $\mathcal{S}$ by construction. For the second containment, it suffices to check that for each $1 \leq i \leq m$ and $1 \leq k \leq p$,
$$y_k \in \mathbb{Z}[x_1^{\pm 1}, ..., x_i'^{\pm 1}, ..., x_n^{\pm 1}]$$
This is because $x_1, ..., x_m, x_{m+1}^{\pm 1}, ..., x_n^{\pm 1}$ are in $\mathcal{U}_{\mathbf{x}}$ by the Laurent phenomenon.
- **Check whether $\mathcal{S} = \mathcal{U}$ using Lemma 4.4.2.** Any of the four criteria $(2)-(5)$ in Lemma 4.4.2 can be used. They all may be implemented by a computer, and each method potentially involves a different algorithm, so any of the four might be the most efficient computationally.
- **If $\mathcal{S} \subsetneq \mathcal{U}$, find additional generators and return to the beginning.** If $\mathcal{S} \neq \mathcal{U}$, then $(\mathcal{S}f : (\mathcal{S}\mathbb{D}_{\mathbf{x}})^\infty)f^{-1}$ contains elements of $\mathcal{U}$ which are not in $\mathcal{S}$ (where $f = x_1x_2...x_m$). One or more of these elements may be added to the original list of Laurent polynomials to get a larger guess S' for $\mathcal{U}$. Note that any S' produced this way satisfies $L_{\mathbf{x}} \subseteq S' \subseteq \mathcal{U}_{\mathbf{x}}$.

5.2. **An iterative algorithm.** The preceeding steps can be regarded as an iterative algorithm for producing successively larger subrings $S \subseteq \mathcal{U}$ as follows. Start with an initial guess $L_{\mathbf{x}} \subseteq S \subseteq \mathcal{U}_{\mathbf{x}}$. In lieu of cleverness, the lower bound $L_{\mathbf{x}} = S$ makes an functional initial guess; this amounts to starting with generators $x_1, ..., x_m, x'_1, ..., x'_m$.

Denote $S_1 := S$, and inductively define S_{i+1} to be the sub-$\mathbb{ZP}$-algebra of $\mathbb{Q}(x_1, x_2, ..., x_n)$ generated by S_i and $(S_if : (S_iI)^\infty)f^{-1}$. If S_i is finitely generated over $\mathbb{ZP}$ (resp. Noetherian), then the saturation $(S_if : (S_iI)^\infty)$ is finitely generated over S_i and so S_{i+1} is finitely generated over $\mathbb{ZP}$ (resp. Noetherian).

This gives a nested sequence of subrings
$$L_{\mathbf{x}} \subseteq S = S_1 \subseteq S_2 \subseteq S_3 \subseteq ... \subseteq \mathcal{U} = \mathcal{U}_{\mathbf{x}}$$
By Lemma 4.2.1, if $S_i = S_{i+1}$, then $S_i = S_{i+1} = S_{i+2} = ... = \mathcal{U} = \mathcal{U}_{\mathbf{x}}$.

Proposition 5.2.1. *If $\mathcal{U}$ is finitely generated over S, then for some i, $S_i = \mathcal{U}$.*

Proof. Let $f = x_1x_2...x_m$. By Proposition 4.1.2,
$$(S_if : (S_i\mathbb{D}_{\mathbf{x}})^\infty) = S_if^{-1} \cap \mathcal{U}$$
Induction on i shows that $Sf^{-i} \cap \mathcal{U} \subseteq S_{i+1}$. If $\mathcal{U}$ is finitely generated over S, then there is some i such that Sf^{-i+1} contains a generating set, and so $S_i = \mathcal{U}$. □

Corollary 5.2.2. *Let be $\mathcal{A}$ a totally coprime cluster algebra, and $S = L_{\mathbf{x}}$ for some seed in $\mathcal{A}$. If $\mathcal{U}$ is finitely generated, then $\mathcal{U} = S_i$ for some i.*

In other words, this algorithm will always produce $\mathcal{U}$ in finitely many steps, even starting with the 'worst' guess $L_{\mathbf{x}}$.

Remark 5.2.3. This algorithm can be implemented by computational algebra software, so long as the initial guess S is finitely presented. However, in the authors' experience, naively implementing this algorithm was computationally prohibitive after the first step. A more effective approach was to pick a few simple elements of $(S_if : (S_iI)^\infty)$ and use them to generate a bigger ring S_{i+1}.

6. Examples: $m = n = 3$

The smallest non-acyclic seed will have $m = n = 3$; that is, 3 mutable variables and no frozen variables. We consider these examples.

6.1. **Generalities.** Consider an arbitrary skew-symmetric seed $(\mathbf{x}, \mathsf{B}_{a,b,c})$ with $m = n = 3$, as in Figure 3. Let $\mathcal{A}_{a,b,c}$ and $\mathcal{U}_{a,b,c}$ be the corresponding cluster algebra and upper cluster algebra, respectively.[9]

$$\mathsf{B}_{a,b,c} = \begin{bmatrix} 0 & -a & c \\ a & 0 & -b \\ -c & b & 0 \end{bmatrix}$$

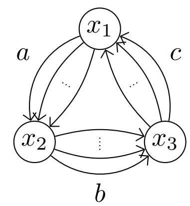
$\mathsf{Q}_{a,b,c} =$ 

FIGURE 3. A general skew-symmetric seed with 3 mutable variables

The seed $(\mathbf{x}, \mathsf{B}_{a,b,c})$ is acyclic unless $a, b, c > 0$ or $a, b, c < 0$, and permuting the variables can exchange these two inequalities. Even when $a, b, c > 0$, the cluster algebra $\mathcal{A}_{a,b,c}$ may not be acyclic, since there may be a acyclic seed mutation equivalent to $(\mathbf{x}, \mathsf{B}_{a,b,c})$. Thankfully, there is a simple inequality which detects when $\mathcal{A}_{a,b,c}$ is acyclic.

Theorem 6.1.1. [BBH11, Theorem 1.1] *Let $a, b, c > 0$. The seed $(\mathbf{x}, \mathsf{B}_{a,b,c})$ is mutation-equivalent to an acyclic seed if and only if $a < 2$, $b < 2$, $c < 2$, or*

$$abc - a^2 - b^2 - c^2 + 4 < 0$$

Acyclic $\mathcal{A}_{a,b,c} = \mathcal{U}_{a,b,c}$ can be presented using [BFZ05, Corollary 1.21]; and so we focus on the non-acyclic cases. As the next proposition shows, these cluster algebras are totally coprime, and so it will suffice to present $\mathcal{U}_{\mathbf{x}}$.

Proposition 6.1.2. *Let $\mathcal{A}$ be a cluster algebra with $m = 3$. If $\mathcal{A}$ is not acyclic, then $\mathcal{A}$ is totally coprime.*

Proof. Let $(\mathbf{x}, \mathsf{B})$ be a non-acyclic seed for $\mathcal{A}$ with quiver Q; that is, there is a directed cycle of mutable cluster variables. There are no 2-cycles in Q by construction, and so the directed cycle in Q passes through every vertex. It follows that $\mathsf{B}_{ij} \neq 0$ if $i \neq j$. Then the ith and jth columns are linearly independent, because $\mathsf{B}_{ii} = 0$ and $\mathsf{B}_{ij} \neq 0$. Hence, $(\mathbf{x}, \mathsf{B})$ is a coprime seed, and $\mathcal{A}$ is totally coprime. □

Remark 6.1.3. This proof does not assume that B^0 is skew-symmetric or that $n = 3$ (ie, that there are no frozen variables).

6.2. **The (a, a, a) cluster algebra.** Consider $a = b = c \geq 0$ as in Figure 4.

$$\mathsf{B}_{a,a,a} = \begin{bmatrix} 0 & -a & a \\ a & 0 & -a \\ -a & a & 0 \end{bmatrix}$$

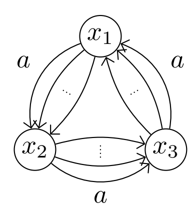
$\mathsf{Q}_{a,a,a} =$ 

FIGURE 4. The exchange matrix and quiver for the (a, a, a) cluster algebra

If $a = 0$ or 1, then $\mathcal{A}_{a,a,a}$ is acyclic.[10] For $a \geq 2$, $\mathcal{A}_{a,a,a}$ is not acyclic by Theorem 6.1.1.

[9]The notation $\mathcal{U}_{a,b,c}$ is dangerous, in that it leaves no room to distinguish between the upper cluster algebra and the upper bound of $\mathsf{B}_{a,b,c}$. However, we will only consider non-acyclic examples, and so by Theorem 2.4.2, these two algebras coincide. The reader is nevertheless warned.

[10]In fact, finite-type of type $A_1 \times A_1 \times A_1$ or A_3, respectively.

Remark 6.2.1. The case $a = 2$ was specifically investigated in [BFZ05], as the first example of a cluster algebra for which $\mathcal{A} \neq \mathcal{U}$, and it has been subsequently connected to the Teichmüller space of the the once-punctured torus and to the theory of Markov triples [FG07, Appendix B] ($\mathcal{A}_{2,2,2}$ is sometimes called the *Markov cluster algebra*). See Section 7.1 for the analog of $\mathcal{U}_{2,2,2}$ with a specific choice of frozen variables.

Proposition 6.2.2. *For $a \geq 2$, the upper cluster algebra $\mathcal{U}_{a,a,a}$ is generated over $\mathbb{Z}$ by*

$$x_1, x_2, x_3, M := \frac{x_1^a + x_2^a + x_3^a}{x_1 x_2 x_3}$$

The ideal of relations among these generators is generated by

$$x_1 x_2 x_3 M - x_1^a - x_2^a - x_3^a = 0$$

Proof. Since $a^3 - 3a^2 + 4 \geq 0$ for $a \geq 2$, Theorem 6.1.1 implies that this cluster algebra is not acyclic, and Proposition 6.1.2 implies that it is totally coprime.

The element $x_1x_2x_3M - x_1^a - x_2^a - x_3^a$ in $\mathbb{Z}[x_1, x_2, x_3, M]$ is irreducible. The ideal it generates is prime and therefore it is saturated with respect to $x_1x_2x_3$. By Lemma 5.1.1,

$$\mathcal{S} = \mathbb{Z}[x_1, x_2, x_3, M] / < x_1x_2x_3M - x_1^a - x_2^a - x_3^a >$$

is the subring of $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$ generated by x_1, x_2, x_3 and M.

The following identities imply that $L_{\mathbf{x}} \subset \mathcal{S}$.

$$x_1' = x_2x_3M - x_1^{a-1}, \quad x_2' = x_1x_3M - x_2^{a-1}, \quad x_3' = x_1x_2M - x_3^{a-1}$$

The following identities imply that $\mathcal{S} \subset \mathcal{U}_{\mathbf{x}}$.

$$M = \frac{x_1'^{a+1} + (x_2^a + x_3^a)^a}{x_1'^a x_2 x_3} = \frac{x_2'^{a+1} + (x_1^a + x_3^a)^a}{x_1 x_2'^a x_3} = \frac{x_3'^{a+1} + (x_1^a + x_2^a)^a}{x_1 x_2 x_3'^a}$$

Since $\mathcal{S}$ is a hypersurface, it is a complete intersection, and so it Cohen-Macaulay [Eis95, Prop. 18.13], and in particular it is $S2$.[11]

Let P be a prime ideal in $\mathcal{S}$ containing

$$\mathbb{D}_{\mathbf{x}} = < x_1x_2x_3, x_1'x_2x_3, x_1x_2'x_3, x_1x_2x_3' >$$

Since $x_1x_2x_3 \subset P$, at least one of $\{x_1, x_2, x_3\} \in P$ by primality. If any two x_i, x_j are, then

$$x_k^a = x_ix_jx_kM - x_i^a - x_j^a \in P \Rightarrow x_k \in P$$

If only one $x_i \in P$, then $x_i'x_jx_k \in P$ implies that $x_i' \in P$. Then $x_i + x_i'^a = x_jx_kM \in P$, which implies $M \in P$. Additionally, $x_j^a + x_k^a = x_ix_jx_kM - x_i^a \in P$.

Therefore, P contains at least one of the four prime ideals

$$< x_1, x_2, x_3 >, < x_1, x_2^a + x_3^a, M >, < x_2, x_1^a + x_3^a, M >, < x_3, x_1^a + x_2^a, M > \tag{6.1}$$

Since $\{x_1, x_2\}$, $\{x_1, M\}$, $\{x_2, M\}$, and $\{x_3, M\}$ are each regular sequences in S, it follows that $\operatorname{codim}(\mathbb{D}_{\mathbf{x}}) \geq 2$. By Lemma 4.4.2, $\mathcal{S} = \mathcal{U}$. □

Remark 6.2.3. The final step of the proof has some interesting geometric content. In this case, $\mathbb{D} = \mathbb{D}_{\mathbf{x}}$, and the four prime ideals (6.1) are the minimal primes containing $\mathbb{D}$. Geometrically, they define the irreducible components of $V(\mathbb{D})$; that is, the complement of the cluster tori.

One of these components ($x_1 = x_2 = x_3 = 0$) is an affine line on which every cluster variable vanishes. The other 3 components ($x_i = x_j^a + x_k^a = M = 0$) are geometrically reducible; over $\mathbb{C}$ they each decompose into a-many affine lines. Over $\mathbb{C}$, $V(\mathbb{D})$ consists of

[11] A ring is Cohen-Macaulay if and only if it satisfies the Sn property for every n.

$3a+1$-many affine lines, which intersect at the point $x_1 = x_2 = x_3 = M = 0$ and nowhere else.

6.3. **The $(3,3,2)$ cluster algebra.** Consider the initial seed in Figure 5. The cluster algebra $\mathcal{A}_{3,3,2}$ is non-acyclic, by Theorem 6.1.1. Up to permuting the vertices, it is the only non-acyclic $\mathcal{A}_{a,b,c}$ with $0 \leq a,b,c \leq 3$ besides $\mathcal{A}_{2,2,2}$ and $\mathcal{A}_{3,3,3}$.

$$\mathsf{B} = \begin{bmatrix} 0 & -3 & 2 \\ 3 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$$

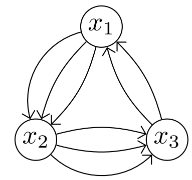
$\mathsf{Q} =$ 

FIGURE 5. The exchange matrix and quiver for the $(3,3,2)$ cluster algebra.

Proposition 6.3.1. *The upper cluster algebra $\mathcal{U}_{3,3,2}$ is generated over $\mathbb{Z}$ by*

$$x_1, x_2, x_3,$$
$$y_0 = \frac{x_2^3 + x_1^2 + x_3^2}{x_1 x_3}, \quad y_1 = \frac{x_1 x_2^3 + x_2^3 x_3 + x_1^3 + x_3^3}{x_1 x_2 x_3},$$
$$y_2 = \frac{x_2^6 + 2x_1^2 x_2^3 + x_1 x_2^3 x_3 + 2x_2^3 x_3^2 + x_1^4 + x_1^3 x_3 + x_1 x_3^3 + x_3^4}{x_1^2 x_2 x_3^2},$$
$$y_3 = \frac{x_2^9 + 3x_1^2 x_2^6 + 3x_2^6 x_3^2 + 3x_1^4 x_2^3 + 3x_1^2 x_2^3 x_3^2 + 3x_2^3 x_3^4 + x_1^6 + 2x_1^3 x_3^3 + x_3^6}{x_1^3 x_2^2 x_3^3}.$$

The ideal of relations is generated by the elements

$$y_2^2 = y_0 y_3 + 2y_3, \quad y_0^2 = x_2 y_2 - y_0 + 2$$
$$y_1 y_2 = x_1 y_3 + x_3 y_3, \quad y_0 y_2 = x_2 y_3 + y_2$$
$$y_0 y_1 = x_1 y_2 + x_3 y_2 - 2y_1, \quad x_1 y_0 + x_3 y_0 = x_2 y_1 + x_1 + x_3$$
$$x_2^2 y_2 = x_1 x_3 y_3 + 3x_2 y_0 - y_1^2 - 3x_2, \quad x_2^2 y_0 = x_1 x_3 y_2 + x_2^2 - x_1 y_1 - x_3 y_1$$
$$x_2^3 + x_3^2 y_0 = x_2 x_3 y_1 - x_1^2 + x_1 x_3.$$

Proof. Since $a = 3, b = 3, c = 2$, and $abc - a^2 - b^2 - c^2 + 4 = 0$, Theorem 6.1.1 implies that $\mathcal{A}$ is not acyclic. Thus, Proposition 6.1.2 asserts that $\mathcal{A}$ is totally coprime. Let $\mathcal{S}$ be the domain in $\mathcal{F}(\mathcal{A})$ generated by the seven listed elements. Using Lemma 5.1.1 and a computer, we see that the ideal of relations in $\mathcal{S}$ is generated by the elements above.

The following identities imply that $L_{\mathbf{x}} \subseteq \mathcal{S}$.

$$x_1' = x_3 y_0 - x_1, \quad x_2' = x_1 x_3 y_1 - x_1 x_2^2 - x_2^2 x_3, \quad x_3' = -x_3 y_0 + x_2 y_1 + x_1$$

The following identities imply that $\mathcal{S} \subseteq \mathcal{U}_{\mathbf{x}}$.

$$y_0 = \frac{x_2^3 + x_3^2 + x_1'^2}{x_3 x_1'} = \frac{(x_1^3 + x_3^3)^3 + (x_1^2 + x_3^2) x_2'^3}{x_1 x_3 x_2'^3} = \frac{x_1^2 + x_2^3 + x_3'^2}{x_1 x_3'}$$

$$\begin{aligned} y_1 &= \frac{(x_2^3 + x_3^2)^2 + x_1'^2 (x_2^3 + x_3 x_1')}{x_2 x_3 x_1'^2} = \frac{(x_1 + x_3)^3 (x_1^2 - x_1 x_3 + x_3^2)^2 + x_2'^3}{x_1 x_3 x_2'^2} \\ &= \frac{(x_1^2 + x_2^3)^2 + x_3'^2 (x_2^3 + x_1 x_3')}{x_1 x_2 x_3'^2} \end{aligned}$$

$$
\begin{aligned}
y_2 &= \frac{(x_2^3+x_3^2)^2 + x_3(x_2^3+x_3^2)x_1' + 2x_2^3x_1'^2 + x_3x_1'^3 + x_1'^4}{x_2x_3^2x_1'^2} \\
&= \frac{(x_1^3+x_3^3)^5 + (2x_1^2+x_1x_3+2x_3^2)(x_1^3+x_3^2)^2x_2'^3 + (x_1+x_3)x_2'^6}{x_1^2x_3^2x_2'^5} \\
&= \frac{(x_1^2+x_2^3-x_1x_3'+x_3'^2)(x_1^2+x_2^3+2x_1x_3'+x_3'^2)}{x_1^2x_2x_3'^2}
\end{aligned}
$$

$$
\begin{aligned}
y_3 &= \frac{(x_2^3+x_3^2-x_3x_1'+x_1'^2)^2(x_2^3+x_3^2+2x_3x_1'+x_1'^2)}{x_2^2x_3^3x_1'^3} \\
&= \frac{((x_1+x_3)^3(x_1^2-x_1x_3+x_3^2)^2+x_2'^3)^2((x_1+x_3)(x_1^2-x_1x_3+x_3^2)^3+x_2'^3)}{x_1^3x_3^3x_2'^7} \\
&= \frac{(x_1^2+x_2^3-x_1x_3'+x_3'^2)^2(x_1^2+x_2^3+2x_1x_3'+x_3'^2)}{x_1^3x_2^2x_3'^3}
\end{aligned}
$$

A computer verifies that $(\mathcal{S}x_1x_2x_3 : (\mathcal{S}\mathbb{D}_{\mathbf{x}})^\infty) = \mathcal{S}x_1x_2x_3$. By Lemma 4.4.2, $\mathcal{S} = \mathcal{U}$. $\square$

Remark 6.3.2. This example serves of a 'proof of concept' for the algorithm of Section 5.2. The above generating set has no distinguishing properties known to the authors; it is merely the generating set produced by an implementation of this algorithm.

7. Larger examples

We explicitly present a few other non-acyclic upper cluster algebras.

7.1. The Markov cluster algebra with principal coefficients.

Consider the initial seed in Figure 6. As in the previous section, this seed has 3 mutable variables, but it has *principal coefficients* – a frozen variable for each mutable variable, and the exchange matrix extended by an identity matrix. Results about principal coefficients and why they are important can be found in [FZ07].

$$
\mathsf{B} = \begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\qquad \mathsf{Q} =
$$

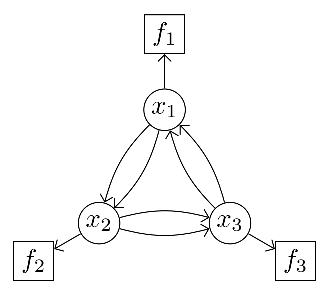


Figure 6. The exchange matrix and quiver for the Markov cluster algebra with principal coefficients.

Proposition 7.1.1. *The upper cluster algebra $\mathcal{U}$ is generated over $\mathbb{Z}[f_1^{\pm1}, f_2^{\pm1}, f_3^{\pm1}]$ by*

$$
x_1, x_2, x_3,
$$

$$
L_1 = \frac{x_2^2 + f_2f_3x_3^2 + f_3x_1^2}{x_2x_3}, L_2 = \frac{x_3^2 + f_3f_1x_1^2 + f_1x_2^2}{x_3x_1}, L_3 = \frac{x_1^2 + f_1f_2x_2^2 + f_2x_3^2}{x_1x_2}
$$

$$
y_1 = \frac{f_1L_1^2 + (f_1f_2f_3-1)^2}{x_1}, y_2 = \frac{f_2L_2^2 + (f_1f_2f_3-1)^2}{x_2}, y_3 = \frac{f_3L_3^2 + (f_1f_2f_3-1)^2}{x_3}
$$

The ideal of relations is generated by the elements

$$
\begin{gathered}
x_1x_2L_3 = x_1^2 + f_1f_2x_2^2 + f_2x_3^2, \quad y_1y_2L_3 = f_1f_2y_1^2 + y_2^2 + f_1y_3^2 \\
x_2x_3L_1 = x_2^2 + f_2f_3x_3^2 + f_3x_1^2, \quad y_2y_3L_1 = f_2f_3y_2^2 + y_3^2 + f_2y_1^2 \\
x_3x_1L_2 = x_3^2 + f_3f_1x_1^2 + f_1x_2^2, \quad y_3y_1L_2 = f_3f_1y_3^2 + y_1^2 + f_3y_2^2 \\
f_3x_1L_3 - x_3L_1 = \alpha x_2, \quad f_1L_1y_3 - L_3y_1 = \alpha y_2 \\
f_1x_2L_1 - x_1L_2 = \alpha x_3, \quad f_2L_2y_1 - L_1y_2 = \alpha y_3 \\
f_2x_3L_2 - x_2L_3 = \alpha x_1, \quad f_3L_3y_2 - L_2y_3 = \alpha y_1 \\
x_1L_2L_3 = f_1f_2x_2L_2 + f_1x_1L_1 + x_3L_3, \quad y_1L_2L_3 = y_2L_2 + f_1y_1L_1 + f_1f_3y_3L_3 \\
x_2L_3L_1 = f_2f_3x_3L_3 + f_2x_2L_2 + x_1L_1, \quad y_2L_3L_1 = y_3L_3 + f_2y_2L_2 + f_2f_1y_1L_1 \\
x_3L_1L_2 = f_3f_1x_1L_1 + f_3x_3L_3 + x_2L_2, \quad y_3L_1L_2 = y_1L_1 + f_3y_3L_3 + f_3f_2y_2L_2 \\
x_2y_3 = f_2f_3L_2L_3 - \alpha L_1, \quad x_3y_1 = f_3f_1L_3L_1 - \alpha L_2, \quad x_1y_2 = f_1f_2L_1L_2 - \alpha L_3 \\
x_1y_3 = L_1L_3 + f_2\alpha L_2, \quad x_2y_1 = L_2L_1 + f_3\alpha L_3, \quad x_3y_2 = L_3L_2 + f_1\alpha L_1 \\
x_1y_1 = f_1L_1^2 + \alpha^2, \quad x_2y_2 = f_2L_2^2 + \alpha^2, \quad x_3y_3 = f_3L_3^2 + \alpha^2 \\
L_1L_2L_3 - f_1L_1^2 - f_2L_2^2 - f_3L_3^2 = \alpha^2
\end{gathered}
$$

where $\alpha := f_1f_2f_3 - 1$.

Proof. The exchange matrix B for the initial seed above contains a submatrix that is a scalar multiple of the identity, thus B is full rank. Theorem 2.4.3 asserts that $\mathcal{A}$ is totally coprime. Let $\mathcal{S}$ be the domain in $\mathcal{F}(\mathcal{A})$ generated by the twelve listed elements. Using Lemma 5.1.1 and a computer, we see that the ideal of relations in $\mathcal{S}$ is generated by the elements above.

The following identities imply that $L_{\mathbf{x}} \subseteq \mathcal{S}$.

$$x_1' = x_3L_2 - f_3f_1x_1, \quad x_2' = x_1L_3 - f_1f_2x_2, \quad x_3' = x_2L_1 - f_2f_3x_3$$

The following identities imply that $\mathcal{S} \subseteq \mathcal{U}_{\mathbf{x}}$.

$$
\begin{aligned}
L_1 &= \frac{x_1'^2x_2^2 + x_1'^2x_3^2f_2f_3 + f_3(x_3^2 + x_2^2f_1)^2}{x_1'^2x_2x_3} = \frac{x_1^2 + x_3^2f_2 + f_3x_2'^2}{x_2'x_3} = \frac{x_3'^2 + f_2f_3(x_2^2 + x_1^2f_3)}{x_2x_3'} \\
L_2 &= \frac{x_2'^2x_3^2 + x_2'^2x_1^2f_3f_1 + f_1(x_1^2 + x_3^2f_2)^2}{x_2'^2x_3x_1} = \frac{x_2^2 + x_1^2f_3 + f_1x_3'^2}{x_3'x_1} = \frac{x_1'^2 + f_3f_1(x_3^2 + x_2^2f_1)}{x_3x_1'} \\
L_3 &= \frac{x_3'^2x_1^2 + x_3'^2x_2^2f_1f_2 + f_2(x_2^2 + x_1^2f_3)^2}{x_3'^2x_1x_2} = \frac{x_3^2 + x_2^2f_1 + f_2x_1'^2}{x_1'x_2} = \frac{x_2'^2 + f_1f_2(x_1^2 + x_3^2f_2)}{x_1x_2'}
\end{aligned}
$$

$$
\begin{aligned}
y_1 &= \tfrac{x_1'^4x_2^2 + x_1'^4x_3^2f_1f_2^2f_3^2 + 2x_1'^2(x_3^2+x_2^2f_1)x_2^2f_1f_3 + f_1f_3^2(x_3^2+x_2^2f_1)^3 + 2x_1'^2(x_3^2+x_2^2f_1)x_3^2f_1f_2f_3^2}{x_1'^2x_2^2x_3^2} \\
&= \frac{f_1(x_1^2 + x_3^2f_2 + f_3x_2'^2)^2 + (f_1f_2f_3 - 1)^2x_2'^2x_3^2}{x_1x_2'^2x_3^2} \\
&= \frac{f_1(x_1(x_3'^3 + x_3'f_2f_3(x_2^2 + x_1^2f_3)))^2 + (f_1f_2f_3 - 1)^2x_1^2x_2^2x_3'^4}{x_1^3x_2^2x_3'^4}
\end{aligned}
$$

$$
\begin{aligned}
y_2 &= \tfrac{x_2'^4x_3^2 + x_2'^4x_1^2f_2f_3^2f_1^2 + 2x_2'^2(x_1^2+x_3^2f_2)x_3^2f_2f_1 + f_2f_1^2(x_1^2+x_3^2f_2)^3 + 2x_2'^2(x_1^2+x_3^2f_2)x_1^2f_2f_3f_1^2}{x_2'^2x_3^2x_1^2} \\
&= \frac{f_2(x_2^2 + x_1^2f_3 + f_1x_3'^2)^2 + (f_2f_3f_1 - 1)^2x_3'^2x_1^2}{x_2x_3'^2x_1^2} \\
&= \frac{f_2(x_2(x_1'^3 + x_1'f_3f_1(x_3^2 + x_2^2f_1)))^2 + (f_2f_3f_1 - 1)^2x_2^2x_3^2x_1'^4}{x_2^3x_3^2x_1'^4}
\end{aligned}
$$

$$\begin{aligned} y_3 &= \frac{x_3'^4x_1^2+x_3'^4x_2^2f_3f_1^2f_2^2+2x_3'^2(x_2^2+x_1^2f_3)x_1^2f_3f_2+f_3f_2^2(x_2^2+x_1^2f_3)^3+2x_3'^2(x_2^2+x_1^2f_3)x_2^2f_3f_1f_2^2}{x_3'^2x_1^2x_2^2} \\ &= \frac{f_3(x_3^2+x_2^2f_1+f_2x_1'^2)^2+(f_3f_1f_2-1)^2x_1'^2x_2^2}{x_3x_1'^2x_2^2} \\ &= \frac{f_3(x_3(x_2'^3+x_2'f_1f_2(x_1^2+x_3^2f_2)))^2+(f_3f_1f_2-1)^2x_3^2x_1^2x_2'^4}{x_3^3x_1^2x_2'^4} \end{aligned}$$

A computer verifies that $(\mathcal{S}x_1x_2x_3 : (\mathcal{S}\mathbb{D}_{\mathbf{x}})^\infty) = \mathcal{S}x_1x_2x_3$. By Lemma 4.4.2, $\mathcal{S} = \mathcal{U}$. □

This presentation is enough to demonstrate an unfortunate pathology of upper cluster algebras. If B is an exchange matrix, and $\mathsf{B}^\dagger$ is an exchange matrix obtained from B by deleting some rows corresponding to frozen variables, then there are natural ring maps

$$s : \mathcal{A}(\mathsf{B}) \to \mathcal{A}(\mathsf{B}^\dagger), \quad s : \mathcal{U}(\mathsf{B}) \to \mathcal{U}(\mathsf{B}^\dagger)$$

which send the deleted frozen variables to 1. It may be naively hoped that the map on upper cluster algebras is a surjection, but this does not always happen.

Corollary 7.1.2. *For B be as in Figure 6 and $\mathsf{B}_{2,2,2}$ as in Figure 3, the map*

$$s : \mathcal{U}(\mathsf{B}) \to \mathcal{U}(\mathsf{B}_{2,2,2})$$

does not contain $M \in \mathcal{U}(\mathsf{B}_{2,2,2})$ in its image, and so it is not surjective.[12]

Proof. One checks $s(f_i) = 1$, $s(x_i) = x_i$, $s(L_i) = x_iM$ and $s(y_i) = x_iM^2$. Let p be the quotient map

$$p : \mathcal{U}(\mathsf{B}_{2,2,2}) \to \mathcal{U}(\mathsf{B}_{2,2,2})/\langle x_1, x_2, x_3\rangle \simeq \mathbb{Z}[p(M)]$$

Since all of the generators of $\mathcal{U}(\mathsf{B})$ map to 0 or 1 under the composition $p \circ s$, $p(M) \in \mathbb{Z}[p(M)]$ is not in the image of $p \circ s$ and so $M \in \mathcal{U}(\mathsf{B}_{2,2,2})$ is not in the image of s. □

7.2. **The 'dreaded torus'.** Consider the initial seed in Figure 7.

$$\mathsf{B} = \begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ -1 & 2 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \qquad \mathsf{Q} =$$

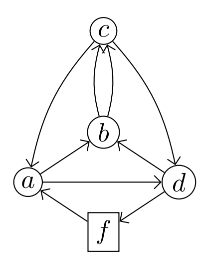


FIGURE 7. The exchange matrix and quiver for the dreaded torus cluster algebra.

Proposition 7.2.1. *The upper cluster algebra $\mathcal{U}$ is generated over $\mathbb{Z}[f^{\pm1}]$ by*

$$a, b, c, d$$

$$X = \frac{b^2+c^2+ad}{bc}, \quad Y = \frac{ad^2+ac^2+bcf+b^2d}{acd}, \quad Z = \frac{a^2d+ac^2+bcf+b^2d}{abd}.$$

The ideal of relations is generated by the elements

$$bcX = b^2+c^2+ad, \quad cY - bZ = d - a$$

$$acX - adZ = ab - bd - cf, \quad bdX - adY = cd - ac - bf$$

$$bXZ - aX - bY - cZ = f.$$

[12]Here, M is defined as in Proposition 6.2.2.

Proof. The exchange matrix B is full rank, and so Theorem 2.4.3 asserts that $\mathcal{A}$ is totally coprime. Let $\mathcal{S}$ be the domain in $\mathcal{F}(\mathcal{A})$ generated by the eight listed elements. Using Lemma 5.1.1 and a computer, we see that the ideal of relations in $\mathcal{S}$ is generated by the elements above.

The following identities imply that $L_{\mathbf{x}} \subseteq \mathcal{S}$.

$$a' = -cX + dZ + b, \quad b' = cX - b, \quad c' = bX - c, \quad d' = -bX + aY + c$$

The following identities imply that $\mathcal{S} \subseteq \mathcal{U}_{\mathbf{x}}$.

$$\begin{aligned} X &= \frac{(b^2+c^2)a' + (bd+cf)d}{a'bc} = \frac{c^2+ad+b'^2}{cb'} \\ &= \frac{(c^2+b^2)d' + (ca+bf)a}{d'cb} = \frac{b^2+da+c'^2}{bc'} \end{aligned}$$

$$\begin{aligned} Y &= \frac{d^2+c^2+a'b}{cd} = \frac{b'^2ad^2 + b'^2ac^2 + b'(c^2+ad)cf + (c^2+ad)^2d}{ab'^2cd} \\ &= \frac{c'^2d + a(b^2+ad) + c'bf}{ac'd} = \frac{a(ac+bf) + d'^2c + d'b^2}{acd'} \end{aligned}$$

$$\begin{aligned} Z &= \frac{a^2+b^2+d'c}{ba} = \frac{c'^2da^2 + c'^2db^2 + c'(b^2+da)bf + (b^2+da)^2a}{dc'^2ba} \\ &= \frac{b'^2a + d(c^2+da) + b'cf}{db'a} = \frac{d(db+cf) + a'^2b + a'c^2}{dba'} \end{aligned}$$

A computer verifies that $(\mathcal{S}abcd : (\mathcal{S}\mathbb{D}_{\mathbf{x}})^\infty) = \mathcal{S}abcd$. By Lemma 4.4.2, $\mathcal{S} = \mathcal{U}$. $\square$

This presentation makes it easy to explore the geometry of $Spec(\mathcal{U})$. One interesting result is the following, which can be proven by computer verification.

Proposition 7.2.2. *The induced deep ideal $\mathcal{U}\mathbb{D}$ is trivial.*

As a consequence, $Spec(\mathcal{U})$ is covered by the cluster tori $\{Spec(\mathbb{Z}[x_1^{\pm1}, ..., x_n^{\pm1}])\}$ coming from the clusters of $\mathcal{U}$. Since affine schemes are always quasi-compact,[13] this cover has a finite subcover; that is, some finite collection of cluster tori cover $Spec(\mathcal{U})$.

Remark 7.2.3. This cluster algebra comes from a marked surface S with boundary (via the construction of [FST08]); specifically, S is the torus with one boundary component and a marked point on that boundary component.

A marked surface also determines a (commutative) *skein algebra* $Sk_1(S)$ defined topologically in terms of immersed curves in S. In [Mul12], it was shown that a certain localization $Sk_1^o(S)$ of $Sk_1(S)$ fit naturally into containments $\mathcal{A}(S) \subseteq Sk_1^o(\mathcal{S}) \subseteq \mathcal{U}(\mathcal{S})$. In this perspective, the generators X, Y, Z of $\mathcal{U}(S)$ correspond to loops in $Sk_1(S)$. As a consequence, $Sk_1^o(S)$ contains a generating set for $\mathcal{U}(S)$, and so the two algebras coincide.

Remark 7.2.4. The epithet 'the dreaded torus' was coined by Gregg Musiker in a moment of frustation – among cluster algebras of surfaces, it lies in the grey area between having enough marked points to be provably well-behaved [Mul13, MSW11] and having few enough marked points to be provably poorly-behaved (like the Markov cluster algebra). For example, it is still not clear whether $\mathcal{A} = \mathcal{U}$ in this case (despite the presentation for $\mathcal{U}$ above).

[13][Har77, Exercise 2.13.b].

8. Acknowledgements

This paper would not have been possible without helpful insight from S. Fomin, J. Rajchgot, D. Speyer, and K. Smith. This paper owes its existence to the VIR seminars in Cluster Algebras at LSU, and to the second author's time at MSRI's Thematic program on Cluster Algebras.

References

[BBH11] Andre Beineke, Thomas Brüstle, and Lutz Hille, *Cluster-cylic quivers with three vertices and the Markov equation*, Algebr. Represent. Theory **14** (2011), no. 1, 97–112, With an appendix by Otto Kerner. MR 2763295 (2012a:16028)

[BFZ05] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky, *Cluster algebras. III. Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), no. 1, 1–52. MR 2110627 (2005i:16065)

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. MR MR1322960 (97a:13001)

[FG06] Vladimir Fock and Alexander Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. (2006), no. 103, 1–211. MR 2233852 (2009k:32011)

[FG07] Vladimir V. Fock and Alexander B. Goncharov, *Dual Teichmüller and lamination spaces*, Handbook of Teichmüller theory. Vol. I, IRMA Lect. Math. Theor. Phys., vol. 11, Eur. Math. Soc., Zürich, 2007, pp. 647–684. MR 2349682 (2008k:32033)

[FP13] Sergey Fomin and Pavlo Pylyavsky, *Tensor diagrams and cluster algebras*, preprint, arxiv: 1210.1888 (2013).

[FST08] Sergey Fomin, Michael Shapiro, and Dylan Thurston, *Cluster algebras and triangulated surfaces. I. Cluster complexes*, Acta Math. **201** (2008), no. 1, 83–146. MR 2448067 (2010b:57032)

[FZ02] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529 (electronic). MR 1887642 (2003f:16050)

[FZ03] ———, *Cluster algebras. II. Finite type classification*, Invent. Math. **154** (2003), no. 1, 63–121. MR 2004457 (2004m:17011)

[FZ07] ———, *Cluster algebras. IV. Coefficients*, Compos. Math. **143** (2007), no. 1, 112–164. MR 2295199 (2008d:16049)

[GLS08] Christof Geiss, Bernard Leclerc, and Jan Schröer, *Partial flag varieties and preprojective algebras*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 3, 825–876. MR 2427512 (2009f:14104)

[GSV03] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein, *Cluster algebras and Poisson geometry*, Mosc. Math. J. **3** (2003), no. 3, 899–934, 1199, {Dedicated to Vladimir Igorevich Arnold on the occasion of his 65th birthday}. MR 2078567 (2005i:53104)

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR MR0463157 (57 #3116)

[MSW11] Gregg Musiker, Ralf Schiffler, and Lauren Williams, *Positivity for cluster algebras from surfaces*, Adv. Math. **227** (2011), no. 6, 2241–2308. MR 2807089

[Mul12] G. Muller, *Skein algebras and cluster algebras of marked surfaces*, preprint, arxiv: 1204.0020 (2012).

[Mul13] Greg Muller, *Locally acyclic cluster algebras*, Adv. Math. **233** (2013), 207–247. MR 2995670

[Sco06] Joshua S. Scott, *Grassmannians and cluster algebras*, Proc. London Math. Soc. (3) **92** (2006), no. 2, 345–380. MR 2205721 (2007e:14078)

[Spe13] David Speyer, *An infinitely generated upper cluster algebra.*, preprint, arxiv: 1305.6867 (2013).

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70808, USA
E-mail address: `jmath34@lsu.edu`

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70808, USA
E-mail address: `gmuller@lsu.edu`