CHROMATIC SYMMETRIC FUNCTIONS OF CLAW-FREE GRAPHS ARE NOT SCHUR POSITIVE

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ABSTRACT. Chromatic symmetric functions are well-studied symmetric functions in algebraic combinatorics that generalize chromatic polynomials of graphs. In 1995, Stanley introduced these symmetric functions and conjectured that they are Schur positive for claw-free graphs. We give examples of two line graphs, which are thus claw-free, whose chromatic symmetric functions have a negative coefficient in their Schur expansion. We also give a counterexample to the 2018 conjecture of Monical that Schur positive chromatic symmetric functions have saturated Newton polytope when expanded in any finite number of variables. Both of these examples were found using ChatGPT-5.6 Sol Pro.

1. INTRODUCTION

Given a simple graph G with vertices $V(G) = \{1, 2, \ldots, n\}$, the *chromatic symmetric function* of G, denoted by $X_G(\mathbf{x})$, is a symmetric function introduced by Stanley in [Sta95] that generalizes the chromatic polynomial of a graph. The chromatic symmetric function of G is defined as follows:

$$X_G(\mathbf{x}) = \sum_{\kappa} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)},$$

where the sum is over proper colorings $\kappa \colon V(G) \to \mathbb{N}$ of G.

There are important open questions about the chromatic symmetric function $X_G(\mathbf{x})$ of a graph G. For instance, Stanley conjectured in [Sta98, Conjecture 1.4] that $X_G(\mathbf{x})$ expands positively in the Schur function basis (i.e. is s-positive) whenever G is claw-free. Recall that a graph is claw-free if it has no induced subgraph isomorphic to the claw graph $K_{1,3}$.

Conjecture 1.1 (Stanley [Sta95])**.** If G is claw-free, then $X_G(\mathbf{x})$ is s-positive.

Stanley checked that the claw $K_{1,3}$ is not s-positive and verified this conjecture for *co-bipartite graphs*, which are complements of bipartite graphs. Gasharov proved it for claw-free incomparability graphs of posets [Gas96]. In fact, the symmetric functions in this case expand positively in the basis of elementary symmetric functions (i.e. are e-positive), a stronger fact conjectured by Stanley and Stembridge in 1993 [SS93], reduced to the chordal case by Guay-Paquet in 2013 [GP13], and verified by Hikita in 2024 via a probabilistic argument [Hik24]. See also another later proof by Griffin–Mellit–Romero–Weigl–Wen via operators on Macdonald polynomials [GMR+25]. For the case of claw-free and chordal incomparability graph, there are other proofs of s-positivity using representation theory of the symmetric group [BC18, GP16, PS25] and traces of Hecke algebras [Hai93, CHSS16]. In contrast, Dahlberg–Foley–van Willigenburg found families of graphs avoiding claws in a different sense with non e-positive chromatic symmetric functions [DFvW20]. Additional evidence Conjecture 1.1 is the proof of real-rootedness of the independence polynomial of claw-free graphs by Chudnovsky and Seymour [CS07], which Stanley showed would be a corollary if the conjecture holds. On the other hand, s-positivity of symmetric functions is in general a rare phenomenon [Pat19].

A related conjecture is one due to Monical [Mon18, Conjecture 2.37], which we now describe. The Newton polytope of a polynomial $p(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_k]$ is the convex hull in $\mathbb{R}^k$ of the support of p. The polynomial p is said to be SNP (or have saturated Newton polytope) if its Newton polytope is saturated, i.e. if the support of p is equal to the set of lattice points in the Newton polytope of p [MTY19]. The study of the SNP property of chromatic symmetric polynomials $X_G(x_1, \ldots, x_k)$ started with work of Monical [Mon18] and Adve–Robichaux–Yong [ARY20, ARY21].

Conjecture 1.2 (Monical [Mon18])**.** If X_G is s-positive, then $X_G(x_1, \ldots, x_k)$ is SNP for any k.

Already for the claw $K_{1,3}$, we have that $X_{K_{1,3}}(x_1, x_2) = x_1^3 x_2 + x_2^3 x_1$ is not SNP (since $(1, 3)$ and $(3, 1)$ are in the support but not the average $(2, 2)$); however, $X_{K_{1,3}}$ is not s-positive. In [MMS24, Theorem 5.8], the authors, together with Selover, verified Monical's conjecture for claw-free incomparability graphs and showed in [MMS24, Example 7.5] that a stronger notion than the SNP property called *M-convexity* does not hold for s-positive chromatic symmetric functions. We also showed that in order to test Monical's conjecture one needs to consider graphs with $n \geqslant 12$ vertices (see [MMS24, Section 7.4]). This makes it hard to find a counterexample for Conjecture 1.2.

Adve–Robichaux–Yong used computational complexity arguments of coloring of claw-free graphs in [ARY20, Section 1.2] to reveal a tension between Conjectures 1.1 and 1.2. Namely, it is unlikely that both conjectures hold. In this note, we show that there is actually no such tension since, as stated, both conjectures are false. In Section 2, we give examples of two line graphs (all line graphs are claw-free) with 12 vertices and 21 and 22 edges respectively (see Figure 1) whose chromatic symmetric functions are not s-positive. In Section 3, we exhibit a bipartite graph with 12 vertices and 19 edges (see Figure 2) with s-positive chromatic symmetric function and unsaturated Newton polytope. All three of these graphs were found using ChatGPT-5.6 Sol Pro.

1.1. **Related work.** The counterexamples (Example 2.1 and Example 2.2) to Conjecture 1.1 were discovered independently at approximately the same time by Jitendra Prajapati in [Pra26b, Pra26a, Kod].

2. Counterexamples to Schur positivity

Given a graph $H = (V(H), E(H))$, the *line graph* of H, which we denote by $L(H)$, is the graph whose vertices are the edges $E(H)$ of H, with two edges in $E(H)$ being adjacent if they share a vertex in $V(H)$. A well-known property of line graphs is that they are claw-free [Wes96, Theorem 7.1.18]. We use the notation of symmetric functions in [Sta99, Chapter 7]. In particular, for a partition λ, we denote by m_λ and s_λ the monomial and Schur symmetric functions which each form a basis of the ring of symmetric functions.

Example 2.1 (First counterexample to Conjecture 1.1). Consider the graph H_1 with vertices in $\{1, 2, \ldots, 10\}$ and with 12 edges:

$$(1, 8), (2, 7), (3, 6), (3, 10), (4, 5), (4, 9), (5, 9), (6, 10), (7, 9), (7, 10), (8, 9), (8, 10).$$

Let $G_1 = L(H_1)$ be the line graph of H_1 which has 12 vertices and 22 edges. See Figure 1:Left. Since G_1 is a line graph, it is claw-free. Using `SageMath` (see code), we have that X_{G_1} is not Schur positive, since

$$[s_{3333}]X_{G_1} = -64.$$

Example 2.2 (Second counterexample to Conjecture 1.1). Consider the graph H_2 with vertices in $\{1, 2, \ldots, 10\}$ and with 12 edges:

$$(1, 8), (1, 9), (2, 7), (3, 6), (3, 10), (4, 5), (4, 9), (5, 9), (6, 10), (7, 9), (7, 10), (8, 10).$$

Let $G_2 = L(H_2)$ be the line graph of H_2 which has 12 vertices and 21 edges. See Figure 1:Right. Since G_2 is a line graph, it is claw-free. Using `SageMath` (see code), we have that X_{G_2} is not Schur positive, since

$$[s_{3333}]X_{G_2} = -40.$$

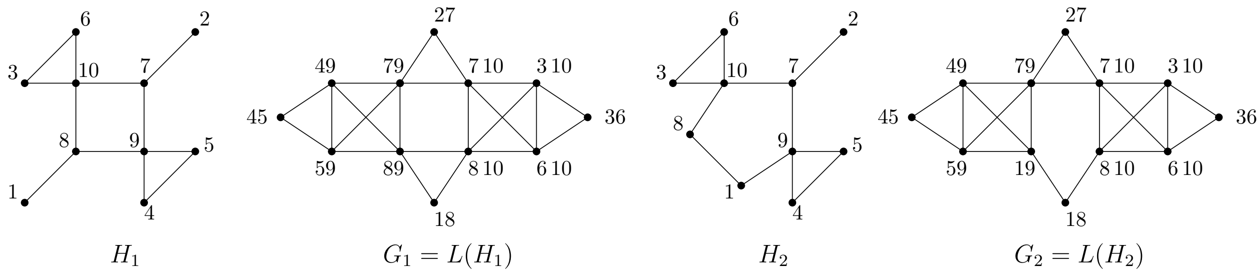

FIGURE 1. Graphs H_i and their line graphs $G_i = L(H_i)$ for $i = 1, 2$. Both G_1 and G_2 are claw-free and have chromatic symmetric functions that are not Schur positive.

Remark 2.3. Since Gasharov proved in [Gas96] that the chromatic symmetric functions of claw-free incomparability graphs are s-positive, it follows that G_1 is not an incomparability graph of a poset. Indeed, one can check this since G_1 has a *net* (three vertices adjacent to a triangle) as a subgraph (e.g., $49, 79, 89, 45, 27, 18$), an obstruction for being a claw-free incomparability graph (see [FHM19, Figure 1]). A similar argument shows that G_2 is also not an incomparability graph of a poset.

Remark 2.4. Prajapati showed in [Pra26b] that up to isomorphism examples G_1 and G_2 are the smallest counterexamples to Conjecture 1.1 with respect to the number of vertices and edges. It would be interesting to find infinite families of connected claw-free graphs with chromatic symmetric functions that are not Schur positive.[1] It would also be interesting to characterize the graphs G such that $X_G(\mathbf{x})$ is s-positive.

3. COUNTEREXAMPLE TO THE SNP PROPERTY

For a coloring $\kappa\colon V(G) \to [k]$, define the *weight* of κ to be

$$\mathrm{wt}(\kappa) = (|\kappa^{-1}(1)|, |\kappa^{-1}(2)|, \ldots, |\kappa^{-1}(k)|) \in \mathbb{N}^k.$$

Thus, the support of $X_G(x_1, \ldots, x_k)$ is the set $\{\mathrm{wt}(\kappa) \mid \kappa\colon V(G) \to [k] \text{ is proper}\}$. We call the vertex sets $\kappa^{-1}(i)$ *color classes*.

Example 3.1 (Counterexample to Conjecture 1.2). Let G_3 be the bipartite graph with vertices $\{1, 2, 9, 10, 11, 12\} \cup \{3, 4, 5, 6, 7, 8\}$ and with 19 edges:

$$\begin{gathered}
(1,3), (1,4), (1,6), (1,7), (1,8) \\
(2,3), (2,4), (2,5), (2,6), (2,7), (2,8) \\
(9,3), (9,4), \quad (10,3), (10,4), \quad (11,3), (11,4) \quad (12,3), (12,4).
\end{gathered}$$

See Figure 2. There are two possible proper colorings with weight $(6, 6)$, namely from the bipartition of the graph: $\kappa^{-1}(i) = \{1, 2, 9, 10, 11, 12\}$ and $\kappa^{-1}(j) = \{3, 4, 5, 6, 7, 8\}$ for $i, j = 1, 2$ and $i, j = 2, 1$, respectively.

There are also two possible proper colorings with weight $(8, 2, 2)$: using one color for the 8 vertices of degree one and two, and a different color for both vertices $\{1, 2\}$, and a third color for both vertices $\{3, 4\}$. See Figure 2:Right.

However, we claim that there is no proper coloring of weight $(7, 4, 1) = \frac{1}{2}((6, 6, 0) + (8, 2, 2))$, the average of the two previous weights. This is because, if there is a proper coloring κ with such a weight then vertices $1, 2, 3, 4$ cannot belong to the color class $\kappa^{-1}(1)$ of size 7 since vertices 2, 3, and 4 are each not adjacent to four vertices, and vertex 1 is not adjacent to six vertices, of which vertices 2 and 5 are adjacent. Thus $\kappa^{-1}(1)$ consist of seven of the eight vertices $5, 6, 7, 8, 9, 10, 11, 12$ (the outer vertices in the drawing in Figure 2).

[1] After this preprint was posted, Wang, Zhang, and Zhao found such an infinite family [WZZ26].

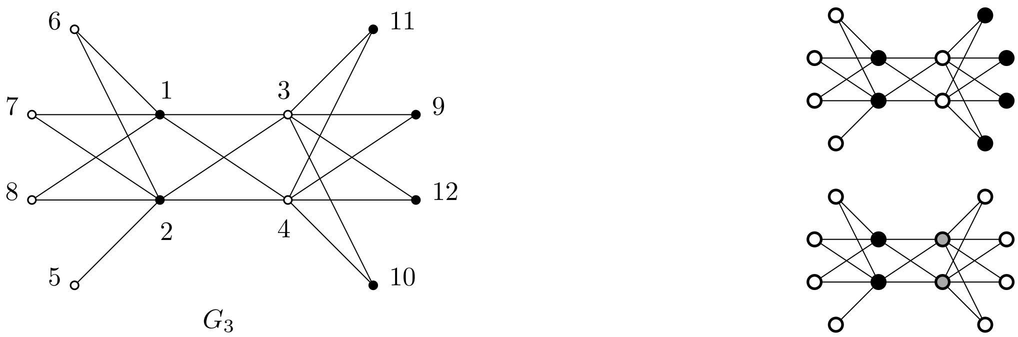

FIGURE 2. Bipartite graph G_3 that has a Schur positive chromatic symmetric function that has a non-saturated Newton polytope, since it has proper colorings of weights $(6,6,0)$ and $(8,2,2)$ (right) but not of the average $(7,4,1)$ of these weights.

Since the vertices $1,2,3,4$ give a copy of $K_{2,2}$, at most two of these vertices have the same color. Thus these vertices and the remaining outer vertex cannot have a weight $(4,1)$, and so there is no such coloring κ.

In the monomial basis, we have shown that $[m_{660}]X_{G_3}(x_1,x_2,x_3) = [m_{822}]X_{G_3}(x_1,x_2,x_3) = 2 \neq 0$, but for the average weight $\frac{1}{2}((6,6,0)+(8,2,2)) = (7,4,1)$ we have that $[m_{741}]X_{G_3}(x_1,x_2,x_3) = 0$. This shows that $X_{G_3}(x_1,x_2,x_3)$ is not SNP.

However, using `SageMath` (see code) we have that $X_{G_3}(\mathbf{x})$ is Schur positive since

$$\begin{gathered}X_{G_2}(\mathbf{x}) = 92528s_{1^{12}}+183128s_{21^{10}}+141902s_{2^21^8}+63544s_{2^31^6}+21768s_{2^41^4}+5348s_{2^51^2}+2612s_{2^6}+154030s_{31^9} \\ +124822s_{321^7}+52422s_{32^21^5}+15824s_{32^31^3}+3319s_{32^41}+29608s_{3^21^6}+16104s_{3^221^4}+4625s_{3^22^21^2}+1311s_{3^22^3} \\ +3018s_{3^31^3}+549s_{3^321}+142s_{3^4}+71908s_{41^8}+49378s_{421^6}+18248s_{42^21^4}+4207s_{42^31^2}+421s_{42^4}+11420s_{431^5} \\ +6278s_{4321^3}+1436s_{432^21}+1101s_{43^21^2}+125s_{43^22}+1272s_{4^21^4}+933s_{4^221^2}+227s_{4^22^2}+113s_{4^231}+28s_{4^3}+20288s_{51^7} \\ +10660s_{521^5}+3293s_{52^21^3}+493s_{52^31}+1979s_{531^4}+1030s_{5321^2}+185s_{532^2}+189s_{53^21}+341s_{541^3}+194s_{5421} \\ +29s_{543}+69s_{5^21^2}+5s_{5^22}+3480s_{61^6}+1291s_{621^4}+321s_{62^21^2}+54s_{62^3}+169s_{631^3}+90s_{6321}+24s_{63^2}+33s_{641^2} \\ +21s_{642}+11s_{651}+2s_{6^2}+336s_{71^5}+82s_{721^3}+22s_{72^21}+6s_{731^2}+6s_{732}+14s_{81^4}+2s_{821^2}+2s_{82^2}.\end{gathered}$$

This shows that the graph G_3 is a counterexample to Monical's conjecture.

Remark 3.2. Note that the graph G_3 in the example above contains claws (e.g., vertices $1,6,7,8$). It would be interesting to see if there is a counterexample to Monical's conjecture that is claw-free.

4. HOW THESE EXAMPLES WERE FOUND

As discussed in the introduction, while writing [MMS24] circa 2021, the authors became interested in the perceived tension between Conjecture 1.1 and Conjecture 1.2. Selover used a SAT solver in some attempts to find a counterexample to Monical's conjecture on line graphs with at least 12 vertices. Since then, the authors of this note identified these two conjectures as amenable to be tested with LLMs. In July 2026, independently and without coordinating, the authors of this note both tried ChatGPT-5.6 Sol Pro on Monical's conjecture. For JPM's session it found counterexample G_3 on the first prompt after 37 minutes (see example of the output here). For AHM's session, it did not find the counterexample after one prompt and around 98 minutes of computation. The second prompt by AHM asked to find a counterexample to Stanley's claw-free conjecture. After 83 minutes it found the two counterexamples G_1 and G_2 in part using SAT solvers (see example of the prompts and output here).

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