Kazhdan–Lusztig polynomials of matroids

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AMS Special Session on Combinatorics and Representation Theory of Reflection Groups: Real and Complex

Outline

1. Review of KL theory for Coxeter groups
2. KL theory for matroids

Coxeter system

Let (W, S) be a Coxeter system. It comes equipped with

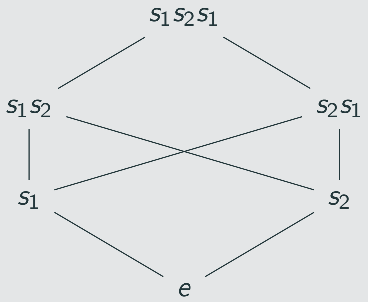
- length function ℓ
- Bruhat order $\leq$.

Bruhat poset

Example

Let $W = \mathfrak{S}_3 = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$

Bruhat poset:



R-polynomials and KL polynomials

First define easier R-polynomials recursively. Let $s \in S$ be such that $ys < y$. Then,

$$R_{x,y}(q) = \begin{cases} 1 & \text{if } x = y \\ R_{xs,ys}(q) & \text{if } xs < x \\ qR_{xs,ys}(q) + (q-1)R_{x,ys}(q) & \text{else.} \end{cases}$$

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Definition (Kazhdan–Lusztig 1979)

To each pair $x, y \in W$, there is a unique polynomial $P_{x,y}(q) \in \mathbb{Z}[q]$ such that

- $P_{x,x}(q) = 1$.
- If $x < y$, then $\deg P_{x,y}(q) \leq \frac{1}{2}(\ell(y) - \ell(x) - 1)$.
- $q^{\frac{1}{2}(\ell(w)-\ell(x))} P_{x,w}(q^{-1}) = q^{\frac{1}{2}(\ell(x)-\ell(w))} \sum_{x \leq y \leq w} R_{x,y} P_{y,w}$.

Which polynomials do we get?

Theorem (Polo 1999)

If p is a polynomial with non-negative integer coefficients and constant term 1, then p is a Kazhdan–Lusztig polynomial for $W = \mathfrak{S}_n$ (for some n).

Schubert varieties

G - algebraic group (connected, reductive over $\mathbb{C}$)

W - its Weyl group

For each $w \in W$, a Schubert variety is a

$$\text{certain subvariety } \overline{X_w} \text{ of the flag variety } G/B.$$

Their intersection cohomology $\mathrm{IH}^\bullet(\overline{X_w})$ is encoded in the $P_{w,y}(q)$.

Basics of intersection cohomology

Basics of intersection cohomology $\mathrm{IH}^\bullet$ of a space:

- gives a measure of the complexity of the singularities of a space
- satisfies Poincaré duality for singular spaces
- isomorphic to singular cohomology for smooth spaces

Intersection cohomology has been a useful tool in geometric representation theory.

Geometric description when W is a Weyl group

Theorem (Kazhdan–Lusztig 1980)

If W is a Weyl group of an algebraic group, then

$$P_{x,y}(q) = \sum_{i \geq 0} q^i \dim \mathrm{IH}^{2i}_{X_y}(\overline{X_x})$$

This implies that the coefficients of $P_{x,y}(q)$ are non-negative.

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Theorem (Elias–Williamson 2014)

For W an arbitrary Coxeter group, the coefficients of $P_{x,y}(q)$ are non-negative.

Comparison

KLoCG	KLoM
Coxeter group	
Weyl group	
Bruhat poset	
R-polynomial	
Hecke algebra	?
Polo	
Schubert variety $\overline{X_x}$	

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Matroids

A **matroid** is a gadget that generalizes the notion of linear (in)dependence in a vector space. It has a

- ground set I (finite set)
- a collection of distinguished subsets (independent sets, bases, closed sets, circuits, ...) satisfying some axioms

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- a graded poset (lattice of flats)
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k - field

I - finite set

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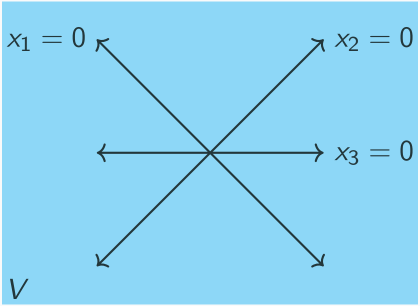
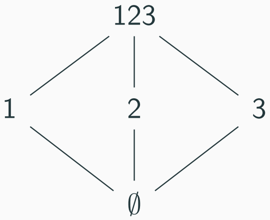
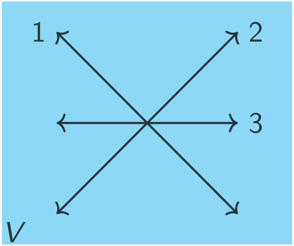
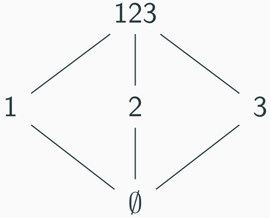
Take $V = (x_1 + x_2 + x_3 = 0) \subset k^{\{1,2,3\}}$. After intersecting with the coordinate hyperplanes, we have





Characteristic polynomial of a matroid





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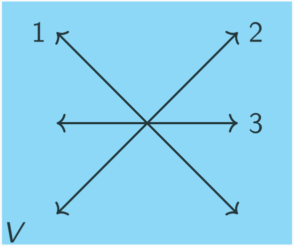

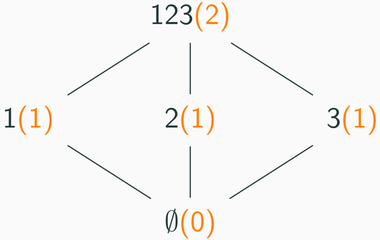

- Rank $\mathrm{rk} : \mathscr{P}(I) \to \mathbb{Z}_{\geq 0}$

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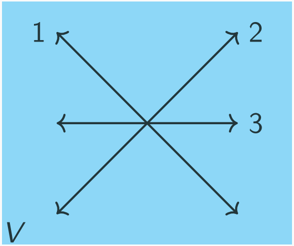

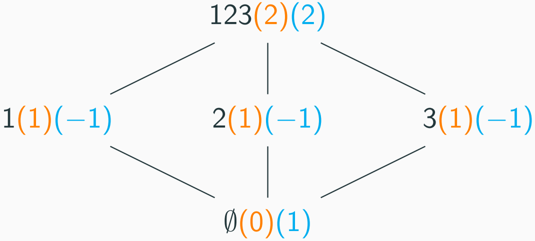

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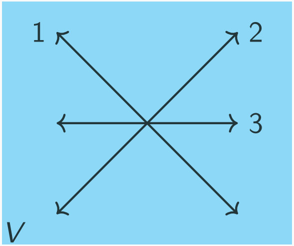

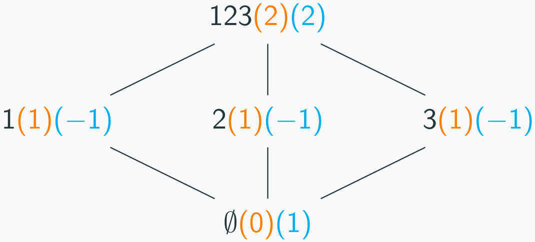

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Definition

The **characteristic polynomial** is

$$\chi_M(q) = \sum_{F \in L(M)} \mu(F) q^{\mathrm{rk} M - \mathrm{rk} F}.$$

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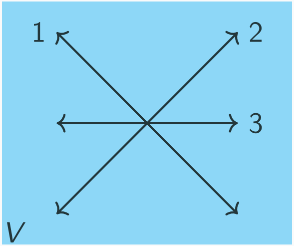

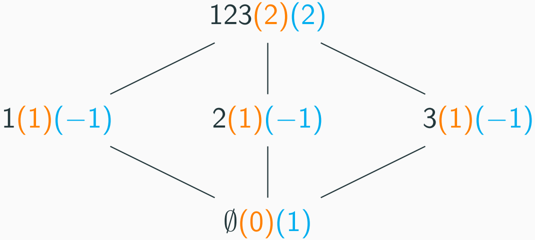

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$$\chi_M(q) = q^2 - 3q + 2$$

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Comparison

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Polo	
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Localization and restriction of matroids

For any flat $F \in L(M)$, we can define two new matroids:

- M_F is a certain matroid on F called the **localization of** M **at** F.
- M^F is a certain matroid on $I \setminus F$ called the **restriction of** M **at** F.

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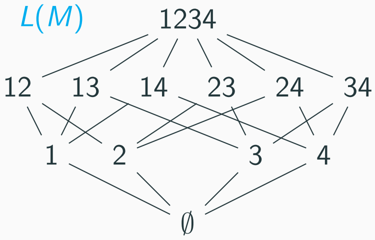

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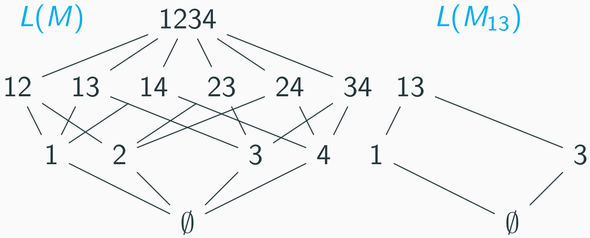

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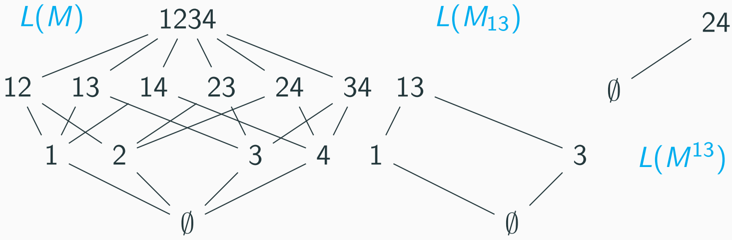

Definition of KL polynomials of matroids

Definition (Elias–Proudfoot–Wakefield 2016)

To each matroid M, we have a unique polynomial $P_M(t) \in \mathbb{Z}[t]$ such that

- If $\mathrm{rk} M = 0$, then $P_M(t) = 1$.
- If $\mathrm{rk} M > 0$, then $\deg P_M(t) < \frac{1}{2}\mathrm{rk} M$.
- For every M, $t^{\mathrm{rk} M} P_M(t^{-1}) = \sum_F \chi_{M_F}(t) P_{M^F}(t)$.

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What do these polynomials look like?

Examples [Elias–Proudfoot–Wakefield–Young 2016]

$M_{m,d}$ is the uniform matroid of rank d on a set of cardinality $m + d$.

Earlier examples: $M_{1,2}$ and $M_{1,3}$

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Kazhdan–Lusztig polynomials for the uniform matroid $M_{1,d}$.

$d =$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
t			2	5	9	14	20	27	35	44	54	65
t^2					5	21	56	120	225	385	616	936
t^3							14	84	300	825	1925	4004
t^4									42	330	1485	5005
t^5											132	1287

Examples [Elias–Proudfoot–Wakefield–Young 2016]

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1	1	1	1	1	1	1	1	1	1	1
t				1	5	16	42	99	219	466
t^2						15	175	1225	6769	32830
t^3								735	16065	204400
t^4										76545

Properties

Conjecture (Elias–Proudfoot–Wakefield 2016)

For any matroid M, the coefficients of $P_M(t)$ are non-negative.

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A sequence $a_0, \ldots, a_r$ is called **log-concave** if for all $1 < i < r$, we have $a_{i-1}a_{i+1} \leq a_i^2$. The sequence has **no internal zeroes** if $\{i \mid a_i \neq 0\}$ is an interval.

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For any matroid M, the coefficients of $P_M(t)$ form a log-concave sequence with no internal zeroes. Moreover, $P_M(t)$ is real-rooted.

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Theorem (Adiprasito–Huh–Katz 2015)

The absolute values of the coefficients of $\chi_M(t)$ form a log-concave sequence with no internal zeroes.

Solved conjectures of Read (1968) and Rota–Heron–Welsh (1970s).

Comparison

KLoCG	KLoM
Coxeter group	matroid
Weyl group	
Bruhat poset	lattice of flats $L(M)$
R-polynomial	characteristic polynomial χ_M
Hecke algebra	?
Polo	real-rooted
Schubert variety $\overline{X_x}$	

Representable matroids

A representable matroid is one that arises as vectors in a vector space.

Theorem (Elias–Proudfoot–Wakefield 2016)

If M is a representable matroid, then

$$P_M(t) = \sum_{i \geq 0} t^i \dim \mathrm{IH}^{2i}(X(V)).$$

Reciprocal plane

Consider the map

$$\begin{array}{ccc} (k^\times)^I & \xrightarrow{\iota} & (k^\times)^I \\ (z_i)_{i\in I} & \longmapsto & (z_i^{-1})_{i\in I}. \end{array}$$

Define the **reciprocal plane** to be

$$X(V) := \overline{\iota(V \cap (k^\times)^I)}.$$

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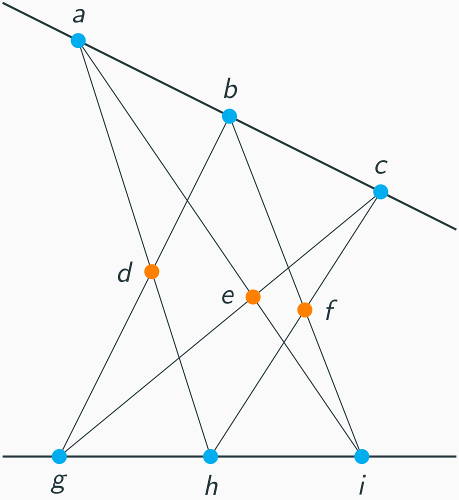
Weyl group	representable matroid
Bruhat poset	lattice of flats $L(M)$
R-polynomial	characteristic polynomial $\chi_M =$
Hecke algebra	?
Polo	real-rooted
Schubert variety $\overline{X_x}$	reciprocal plane $X(V)$

Example of a non-representable matroid



the non-Pappus matroid

Some ideas for non-negativity

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Output: a complex of graded vector spaces $\mathcal{C}^\bullet(M)$

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Conjecture (Braden–M.–Proudfoot)

The complex $\mathcal{C}^\bullet(M)$ has the following properties:

- $H^i(\mathcal{C}^\bullet(M))_j = 0$ *unless* $i = j$, *and*
- $\dim H^i(\mathcal{C}^\bullet(M))_i$ *is the coefficient of* t^i *in* $P_M(t)$.

The complex for IH^2

In this case, $\mathcal{C}^\bullet(M)$ is a two-step complex, and its degree 1 piece is

$$\begin{array}{rcl} \langle \omega_i \rangle_{i\in I} & \xrightarrow{\phi} & \langle 1_E \rangle_{\mathrm{rk} M - \mathrm{rk} E = 1} \\ \omega_i & \longmapsto & \displaystyle\sum_{i \notin E} 1_E . \end{array}$$

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The Euler characteristic of $\mathcal{C}^\bullet(M)$ is the linear coefficient of $P_M(t)$.

The map ϕ is injective, so this coefficient is given by
$\dim H^1(\mathcal{C}^\bullet(M))_1 = \#\text{coatoms} - \#\text{atoms}$.

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Corollary (Elias–Proudfoot–Wakefield 2016, Braden–M.–Proudfoot)

For any matroid M, the linear term of $P_M(t)$ is non-negative.

The complex for IH^4

Theorem (Braden–M.–Proudfoot)

The degree 2 part of our complex $\mathcal{C}^\bullet(M)$ computes $\mathrm{IH}^4(X(V))$.

Corollary (Braden–M.–Proudfoot)

For any matroid M, the quadratic term of $P_M(t)$ is non-negative.

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Thanks!