# GENERALIZED SPLINES ON GRAPHS WITH TWO LABELS AND POLYNOMIAL SPLINES ON CYCLES

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Abstract. A *generalized spline* on a graph $G$ with edges labeled by ideals in a ring $R$ consists of a vertex-labeling by elements of $R$ so that the labels on adjacent vertices $u, v$ differ by an element of the ideal associated to the edge $uv$. We study the $R$-module of generalized splines and produce minimum generating sets for several families of graphs and edge-labelings: 1) for all graphs when the edge-labelings consist of at most two finitely-generated ideals, and 2) for cycles when the edge-labelings consist of principal ideals generated by elements of the form $(ax+by)^2$ in the polynomial ring $\mathbb{C}[x, y]$. We obtain the generators using a constructive algorithm that is suitable for computer implementation and give several applications, including contextualizing several results in classical (analytic) splines.

## 1. Introduction

Splines are a fundamental tool in applied mathematics and analysis, used in fields from data interpolation to computer graphics and design. They are traditionally defined as piecewise polynomials on a combinatorial partition of a geometric object that agree up to some specified differentiability on the intersection of the top-dimensional pieces of the partition. The most common example of these combinatorial partitions in the literature is a polyhedral or simplicial decomposition of a suitable region in Euclidean space.

This paper considers an algebraic generalization of classical splines: given a (combinatorial) graph $G$ with edges labeled by ideals in some fixed ring $R$, a spline is an $R$-labeling of the vertices so that the labels on adjacent vertices $u, v$ differ by an element of the ideal labeling the edge $uv$. This formulation is due to work of the third author with Gilbert and Viel [GTV16], but was first used by Billera [Bil88] and (in the context of equivariant cohomology) by Guillemin–Zara [GZ00, GZ01a, GZ01b]. The construction of generalized splines is essentially dual to the original definition of splines [Bil88, Theorem 2.4]. For example, in the case of a triangulation of a region in the plane, the vertices of $G$ correspond to triangles of the triangulation, and the edge-relations correspond to differentiability conditions across intersections of triangles. Both classical splines and generalized splines are used to construct torus-equivariant cohomology [Pay06, GKM98]. (See Section 5 for more.)

One of the most important problems in the study of splines is to identify the dimension of the spline space, interpreted either as the dimension of the vector space of classical splines of degree at most $d$ [AS87, AS90, Hon91, SSY19, SS02, Sch79, YS19, Str74] (see [LS07] for a survey in the bivariate case) or as the (minimum) number of generators of the module of generalized splines [AS19b, AS19a, BR91, DiP12, GZ01b, GZ03, BHKR15, GTV16, ACFG$^+$20].

In this paper, we compute the number of generators of the module of splines over several families of graphs for different collections of rings. Our most general result is Theorem 3.2, which gives an algorithm using graph connectivity to compute a minimum set of generators for the module of splines over any graph $G$ with exactly two distinct edge labels. The only hypothesis on $R$ is that it

2020 *Mathematics Subject Classification.* 05C25 (Primary); 05C78, 05E16, 41A15 (Secondary).

*Key words and phrases.* generalized splines, minimum generating sets.

Portia Anderson received support from the Smith College Center for Women in Mathematics and from NSF-MTCP-1143716. Jacob Matherne received support from NSF-DMS-1638352, the Association of Members of the Institute for Advanced Study, the Max Planck Institute for Mathematics in Bonn, and the Hausdorff Research Institute for Mathematics in Bonn. Julianna Tymoczko received support from NSF-DMS-1362855 and NSF-DMS-1800773.

be a unique factorization domain (UFD). Theorem 3.1 specializes to the case when all edges of $G$ are labeled by the same ideal; in that case, if the ideal is principal then the module of splines is free over $R$ and its rank is precisely the number of vertices in $G$.

We then specialize $R$ to be a polynomial ring, typically using the assumption that the edge-labels are principal ideals generated by homogeneous splines of the same degree. These assumptions may seem restrictive but are not (see Remark 2.8 and Sections 5.1 and 5.3 for a discussion). Indeed, in all applications that we know, splines use polynomial rings as their base ring (except for certain cases in number theory); furthermore, all known applications use principal ideals as edge-labels. Moreover, the edge-labels used in spline constructions of equivariant cohomology arise as the weights of torus actions on a geometric space, and are naturally homogeneous. Even in cases when the edge-labels are not a priori homogeneous (as in analytic splines), we can homogenize (see Remark 5.10). Corollary 5.9 proves that homogenization induces a natural vector space isomorphism between the classical vector space of splines of degree at most $d$ and the module of splines over the polynomial quotient

$$\mathbb{C}[x_1, x_2, \ldots, x_n]/\langle \text{all monomials in the } x_i \text{ of degree at most } d+1\rangle$$

(considered as a complex vector space). Classical splines do not form a ring since multiplication generally increases degree. However, identifying the vector space of classical splines with the elements of this quotient space allows us to consider a ring structure on splines.

With these assumptions, we prove one other main result. Theorem 4.15 computes explicit (homogeneous) generators for all splines on cycles whose edges are labeled by polynomials of the form $(x+ay)^2$ and shows that these splines cannot be obtained from fewer generators. Indeed, it shows that these generators form a basis. This is a remarkably uniform result that depends only on the number of distinct edge-labels, and not the underlying geometry.

**Corollary 4.16.** *Suppose $C_n$ is a cycle with $n$ vertices and that each edge is labeled by a principal ideal generated by a polynomial of the form $(x+ay)^2$. Then the module of splines has a basis of the following form:*

- *If there is only one distinct edge-label: one homogeneous generator of degree zero and $n-1$ of degree two.*
- *If there are two distinct edge-labels: one homogeneous generator of degree zero, $n-2$ of degree two, and one of degree four.*
- *If there are at least three distinct edge-labels: one homogeneous generator of degree zero, $n-3$ of degree two, and two of degree three.*

We then provide applications related to the *lower bound conjecture* in the classical theory of bivariate splines. The lower bound conjecture arises from an explicit polynomial in $r, d$ that is related to the dimension of the space of splines $S^r_d(\Delta)$ for a large family of triangulations of a given region $\Delta$ of the plane (see Section 5.3 for a precise definition of $S^r_d(\Delta)$). Strang conjectured that this polynomial computes the dimension of $S^r_d(\Delta)$ for specific families of $r$, $d$, and $\Delta$ [Str74]. Schumaker showed that the polynomial provides a lower bound for all $r, d, \Delta$ [Sch79]. Considerable work has happened on this problem since: Alfeld and Schumaker showed that the polynomial gives the dimension when $d \geq 4r+1$ [AS87], which Hong later tightened to $d \geq 3r+2$ [Hon91]; at the same time, Billera proved Strang's conjecture for $r=1$ and $d=3$ as long as the triangulation $\Delta$ is generic [Bil88]. When $r=1$ and $d=3$, the lower bound formula consists of terms contributed by boundary and interior vertices of the triangulation, a correction term for certain interior vertices called "singular vertices," and a constant term from polynomials defined on the entire triangulation (not piecewise). Most mathematicians believe the formula actually computes the dimension of $S^1_3(\Delta)$; there are no known counterexamples to this claim despite significant and ongoing efforts [SS02, YS19, SSY19]. (See [LS07, Chapter 9] for more history and context.)

We give two results that provide a theoretical foundation contextualizing the lower bound formula:

- Theorem 4.15, Lemma 5.13, and Corollary 5.15 give an alternative proof of Schumaker's characterization of splines on a single interior cell (namely "pinwheel triangulations" consisting of a single interior vertex and a number of triangles incident to that vertex and covering a small neighborhood around that vertex) [LS07, Theorems 9.3 and 9.12].
- Corollary 5.16 explains the correction term accounting for "singular vertices" in the lower bound conjecture, as the unique geometrically realizable triangulations that correspond to cycles with exactly two distinct edge-labels.

**Acknowledgments.** Michael DiPasquale's assistance with Macaulay2 [GS] was invaluable to the data collection that seeded this paper. The second author would also like to thank Cleo Roberts and Claudia Yun for helpful discussions about splines.

## 2. Generalized splines on graphs

This section reviews the basic definitions and constructions that we use, including terminology from graph theory and essential results about splines, including the definition of minimum generating sets (MGSs). We state most results in this paper for splines on connected graphs because splines for arbitrary graphs can be obtained from splines on the connected components via direct sum (see Proposition 2.4).

2.1. **Graphs.** For a graph $G = (V, E)$, we denote its (finite) set of vertices by $V$ and its (finite) set of edges by $E$. We write elements of $E$ as pairs of distinct vertices; for example, $e = uv$ is the edge that joins vertex $u$ and vertex $v$, and we say $u$ and $v$ are adjacent. (Note that $uv = vu$ since edges are unoriented.) Graphs in this paper have at most one edge between any given pair of vertices.

If $G' = (V', E')$ is another graph such that $V' \subseteq V$ and $E' \subseteq E$, then $G'$ is called a *subgraph* of $G$. The *induced subgraph* $G[V']$ of $V'$ is the graph with vertex set $V'$ and edge set consisting of all edges in $E$ with both vertices in $V'$. The *neighborhood* $N_G(V')$ of $V'$ is the set of vertices in $V$ that are adjacent to at least one vertex in $V'$. We also define the graph $G - E' := (V, E \setminus E')$.

A *path* in $G$ is a finite sequence of edges $(u_1u_2, u_2u_3, \ldots, u_{n-2}u_{n-1}, u_{n-1}u_n)$ such that each pair of successive edges shares a vertex. A *connected component* of $G$ is a subgraph $G'$ of $G$ with the property that any two vertices of $G'$ are joined by a path lying entirely in $G'$. If $G$ has exactly one connected component, then $G$ is a *connected graph*.

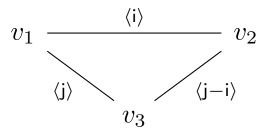
**Proposition 2.1.** *If $G = (V, E)$ is a connected graph, then there is an ordering $v_1, \ldots, v_{|V|}$ on $V$ such that for every $1 < i \leq |V|$ the vertex $v_i$ is adjacent to some vertex $v_j$ with $j < i$.*

*Proof.* We proceed by induction on the number of vertices currently ordered. For the base case, arbitrarily choose a first vertex $v_1 \in V$. Now suppose that we have ordered $v_1, \ldots, v_k$ for some $1 < k < |V|$. The induced subgraph $G[\{v_1, \ldots, v_k\}]$ is connected by the inductive hypothesis and is not all of $G$. If $N_G(\{v_1, \ldots, v_k\})$ were empty, then $G[\{v_1, \ldots, v_k\}]$ would be a connected component of $G$. This contradicts the hypothesis that $G$ is connected, so $N_G(\{v_1, \ldots, v_k\}) \neq \emptyset$. Thus there is some $v_{k+1} \in N_G(\{v_1, \ldots, v_k\})$, and the claim holds by induction. □

2.2. **Splines and minimum generating sets.** Let $R$ be a commutative UFD with identity denoted by 1. Let $\mathcal{I}$ be the set of ideals of $R$. A function $\alpha\colon E \to \mathcal{I}$ is called an *edge-labeling* of $G$. We write $(G, \alpha)$ to mean a graph together with an edge-labeling, and call it an *edge-labeled graph*. Note that, we have suppressed explicit mention of the vertex set and edge set in the notation of an edge-labeled graph, but these will always be clear from context.

**Definition 2.2.** Let $(G, \alpha)$ be an edge-labeled graph. A *spline* on $(G, \alpha)$ is a vertex-labeling $\mathbf{p} \in \bigoplus_{v \in V} R$ that satisfies the GKM condition:

$$\text{for every edge } e = uv \in E, \text{ the difference between coordinates at the endpoints is } \mathbf{p}_u - \mathbf{p}_v \in \alpha(e).$$




$$\mathbf{p} = (0, \mathsf{ij}, \mathsf{j}^2)$$

FIGURE 1. An edge-labeled complete graph on three vertices together with a spline $\mathbf{p}$ on it. Here $\mathsf{i}, \mathsf{j} \in R$.

We sometimes write a spline $\mathbf{p} = (\mathbf{p}_{v_1}, \ldots, \mathbf{p}_{v_{|V|}})$ as a $|V|$-tuple when we have a particular ordering on $V$ in mind. See Figure 1 for an example of a spline on an edge-labeled graph.

**Remark 2.3.** The name "GKM condition" (also appearing in [GTV16]) refers to work by Goresky, Kottwitz, and MacPherson, where this condition appears while combinatorially computing the equivariant cohomology of certain varieties carrying well-behaved torus actions [GKM98].

We write $R_{G,\alpha}$ for the set of splines on the edge-labeled graph $(G, \alpha)$. It is well known that $R_{G,\alpha}$ is itself a ring with identity [GTV16, Proposition 2.4]. The unit $\mathbf{1} \in R_{G,\alpha}$ is given by $\mathbf{1}_v = 1$ for all vertices $v$. (We call $\mathbf{1}$ the *trivial spline*.) Addition and multiplication in $R_{G,\alpha}$ are defined pointwise; that is, $(\mathbf{p}+\mathbf{q})_v = \mathbf{p}_v + \mathbf{q}_v$ and $(\mathbf{pq})_v = \mathbf{p}_v\mathbf{q}_v$ for all $v \in V$. Moreover, $R_{G,\alpha}$ carries the structure of an $R$-module given by $r \cdot \mathbf{p} = (r\mathbf{p}_v)_{v\in V}$ for any $r \in R$.

The following proposition from [GTV16] confirms that our results extend from connected graphs to arbitrary graphs. Recall if $G' = (V', E')$ and $G'' = (V'', E'')$ are graphs, then their *union* is defined as

$$G' \cup G'' = (V' \cup V'', E' \cup E'').$$

**Proposition 2.4** ([GTV16, Proposition 2.11])**.** *Let $(G', \alpha')$ and $(G'', \alpha'')$ be two disjoint edge-labeled graphs, namely $V' \cap V'' = \emptyset$ and $E' \cap E'' = \emptyset$. If $G = G' \cup G''$ and $\alpha$ is the edge-labeling on $G$ defined by restricting to $\alpha'$ on $G'$ and $\alpha''$ on $G''$, then $R_{G,\alpha} = R_{G',\alpha'} \oplus R_{G'',\alpha''}$.*

In this paper, we present algorithms for producing minimum generating sets for $R_{G,\alpha}$ for a variety of edge-labeled graphs $(G, \alpha)$.

**Definition 2.5.** A *generating set* $\mathcal{B}$ for $R_{G,\alpha}$ is a set of splines in $R_{G,\alpha}$ which generates $R_{G,\alpha}$ as an $R$-module. The set $\mathcal{B}$ is called a *minimum generating set* (MGS) if it is a generating set with the property that no other generating set has fewer elements than $\mathcal{B}$.

The general question of when $R_{G,\alpha}$ is a free $R$-module is complicated. In topological applications, $R_{G,\alpha}$ is typically assumed to be free—this is the main implication of *equivariant formality*, which is one of the hypotheses in the machinery of GKM theory (see Section 5.1 or [GKM98, Tym05] for more details). In analytic applications, they need not be (see, for example, [DiP12]). For the most part we do not address this question, though the following lemma applies to several of our results.

**Lemma 2.6.** *Let $R$ be an integral domain and $(G, \alpha)$ be an edge-labeled graph. If $\mathcal{B}$ is an MGS for $R_{G,\alpha}$ that is triangular*[1] *with respect to some vertex ordering $v_1, \ldots, v_{|V|}$ on $V$, then $R_{G,\alpha}$ is a free $R$-module with basis $\mathcal{B}$.*

*Proof.* Since the MGS $\mathcal{B}$ is triangular, it has at most $|V|$ elements and we may order the basis elements $\mathcal{B} = \{\mathbf{b}^{i_1}, \mathbf{b}^{i_2}, \ldots, \mathbf{b}^{i_{k'}}\}$ so that $\mathbf{b}^i_{v_j} = 0$ for all $j < i$ and $\mathbf{b}^i_{v_i} \neq 0$ for all $i \in \{i_1, i_2, \ldots, i_{k'}\}$.

Now suppose $\sum_{\mathbf{b}^i \in \mathcal{B}} c_i \mathbf{b}^i = 0$ is a linear dependence. We prove by induction on $i$ that all $c_i$ are zero—the base case, that $c_{i_1} = 0$, is clear by triangularity. If $c_i = 0$ for all $i < i_0$, then we have

$$\sum_{\mathbf{b}^i \in \mathcal{B}} c_i \mathbf{b}^i_{v_{i_0}} = c_{i_0} \mathbf{b}^{i_0}_{v_{i_0}},$$

[1] An MGS $\mathcal{B}$ is *(upper or lower) triangular* with respect to a vertex ordering $v_1, \ldots, v_{|V|}$ on $V$ if, after ordering the entries of the elements of $\mathcal{B}$ according to the ordering on $V$, the matrix whose columns are the elements of $\mathcal{B}$ is a (upper or lower) triangular matrix with nonzero diagonal entries.

since all $c_i$ with $i < i_0$ are zero by the inductive hypothesis and all $\mathbf{b}^i_{v_{i_0}}$ with $i > i_0$ are zero by triangularity. We assumed the displayed expression was zero, so $c_{i_0}\mathbf{b}^{i_0}_{v_{i_0}} = 0$. But $\mathbf{b}^{i_0}_{v_{i_0}}$ is nonzero by assumption and $R$ is a domain, so $c_{i_0}$ is zero. The claim follows. □

The next result gives a lower bound on the number of elements of an MGS. (This lower bound does not hold if $R$ has zero divisors [BT15].)

**Lemma 2.7.** *If $R$ is an integral domain, then the number of elements of an MGS $\mathcal{B}$ is at least $|V|$.*

*Proof.* If $R$ is an integral domain and $(G, \alpha)$ is a connected edge-labeled graph, then the module of splines $R_{G,\alpha}$ contains a free $R$-submodule $M$ generated by $|V|$ elements [GTV16, Corollary 5.2]. Now consider the image of the $R$-modules $R_{G,\alpha} \supseteq M$ under the map induced by including $R$ into its field of fractions. The image of $M$ is a vector space of dimension $|V|$ and the image of $R_{G,\alpha}$ is a vector space that both contains the image of $M$ and is generated by the image of $\mathcal{B}$. Thus there are at least $|V|$ elements in $\mathcal{B}$. □

**Remark 2.8.** Many of our key results apply to edge-labelings by finitely-generated ideals. However, our results treat principal ideals. We do this for two reasons. First, most applications of splines use edge-labels that are principal ideals (see Section 5 for more). Second, our arguments usually generalize easily to finitely-generated ideals. Indeed, the main step of many of our arguments uses triangular MGSs in which each generator also satisfies $\mathbf{b}^i_u \in \{0, r\}$ for all $u \in V$ and some fixed ring element $r$. In this context, it is straightforward to extend the main results of this paper from edge-labels that are principal ideals to edge-labels that are finitely generated: instead of creating a single spline $\mathbf{b}^i$ for which $\mathbf{b}^i_{v_i}$ generates the principal ideal associated to an edge incident to $v_i$, we create a set of generators $\{\mathbf{b}^{i,1}, \mathbf{b}^{i,2}, \ldots, \mathbf{b}^{i,k}\}$ that minimally generate the ideal associated to that edge. (This kind of argument has been used previously in the literature [HT17, Propositions 2.4 and 2.6].) Expanding the generator set in this fashion gives analogous versions for edge-labelings with finitely-generated ideals of Theorem 3.1, Theorem 3.2, and the dimension computations in Corollary 4.12. We feel that sticking to principal ideals is innocent and leads to improved clarity in the arguments throughout.

## 3. Algorithm to produce an MGS on edge-labeled graphs with one or two labels

In this section, we give an algorithm to produce an MGS for an arbitrary connected graph $G$ whenever the edge-labeling has at most two labels. As a warm-up, in Section 3.1, we treat the case where the image of the edge-labeling $\alpha\colon E \to \mathcal{I}$ is a one-element subset $\{I\}$ of $\mathcal{I}$. In Section 3.2, we treat the case of two edge-labels. Throughout this section, $G$ denotes an arbitrary connected graph.

### 3.1. **One edge-label.**

Let $\alpha\colon E \to \mathcal{I}$ be a constant edge-labeling function; that is, the image of $\alpha$ consists of a single principal ideal $I = \langle \mathsf{i} \rangle$. For a given $v \in V$, denote by $\mathbf{I}^v$ the *indicator spline* of the ideal $I$ at the vertex $v$. In other words, $\mathbf{I}^v$ is the spline with $\mathbf{I}^v_v = \mathsf{i}$ and $\mathbf{I}^v_u = 0$ for all $u \neq v$.

**Theorem 3.1.** *Fix an ordering $v_1, \ldots, v_{|V|}$ on $V$ as in Proposition 2.1, and let $\alpha\colon E \to \mathcal{I}$ be the constant edge-labeling $\alpha(e) = I = \langle \mathsf{i} \rangle \in \mathcal{I}$ for all $e \in E$.*

*Then the set $\mathcal{B} = \{\mathbf{1}, \mathbf{I}^{v_2}, \ldots, \mathbf{I}^{v_{|V|}}\}$ is an MGS for $R_{G,\alpha}$. Moreover, if $R$ is a domain then $R_{G,\alpha}$ is a free $R$-module with basis $\mathcal{B}$.*

*Proof.* Let $\mathbf{p} \in R_{G,\alpha}$ be an arbitrary spline. We claim that there exist $r_2, \ldots, r_{|V|} \in R$ such that

$$\mathbf{p} = \mathbf{p}_{v_1}\mathbf{1} + r_2\mathbf{I}^{v_2} + \cdots + r_{|V|}\mathbf{I}^{v_{|V|}}. \tag{1}$$

We will prove the following statement, which is equivalent to Equation (1): for every $2 \leq j \leq |V|$, there exists $r_j \in R$ such that $\mathbf{p}_{v_j} - \mathbf{p}_{v_1} = r_j\mathsf{i}$. We proceed by induction on $j$.

When $j = 2$, Proposition 2.1 ensures that $v_2$ is adjacent to $v_1$. Thus there exists $r_2 \in R$ such that $\mathbf{p}_{v_2} - \mathbf{p}_{v_1} = r_2\mathsf{i} \in \alpha(v_2v_1)$, as desired.

Our inductive hypothesis states: if $j \in \{2, \ldots, |V|-1\}$, then for all $k$ with $2 \leq k \leq j$ there exists $r_k \in R$ such that $\mathbf{p}_{v_k} - \mathbf{p}_{v_1} = r_k \mathsf{i}$. By Proposition 2.1, the vertex $v_{j+1}$ is adjacent to some $v_\ell$ with $1 \leq \ell \leq j$. If $v_{j+1}$ is adjacent to $v_1$, then the GKM condition ensures that there exists $r_{j+1} \in R$ with $\mathbf{p}_{v_{j+1}} - \mathbf{p}_{v_1} = r_{j+1}\mathsf{i}$ as desired. Otherwise, the spline $\mathbf{p} - \mathbf{p}_{v_1}\mathbf{1} - r_\ell \mathbf{I}^{v_\ell}$ satisfies

$$(\mathbf{p} - \mathbf{p}_{v_1}\mathbf{1} - r_\ell \mathbf{I}^{v_\ell})_u = \begin{cases} 0 & \text{when } u = v_\ell, \\ \mathbf{p}_{v_i} - \mathbf{p}_{v_1} & \text{when } u = v_i \text{ for } i \neq \ell. \end{cases}$$

The GKM condition when $u = v_{j+1}$ implies that there is some $r_{j+1} \in R$ such that $\mathbf{p}_{v_{j+1}} - \mathbf{p}_{v_1} = r_{j+1}\mathsf{i} \in \alpha(v_{j+1}v_\ell)$. Equation (1) follows by induction, so $\mathcal{B}$ is an MGS for $R_{G,\alpha}$ by Lemma 2.7.

Finally, if $R$ is a domain then Lemma 2.6 applies, proving that $R_{G,\alpha}$ is free with basis $\mathcal{B}$. □

3.2. **Two edge-labels.** Now suppose the edge-labeling $\alpha\colon E \to \mathcal{I}$ has image $\{I, J\} \subseteq \mathcal{I}$ with $I = \langle \mathsf{i} \rangle$ and $J = \langle \mathsf{j} \rangle$. The theorem below gives an algorithm for producing an MGS for $R_{G,\alpha}$. The basic idea of the proof is to consider the neighbors of each vertex $v_i$ successively. If $v_i$ is connected to the first $i-1$ vertices only through paths with a single edge-label, then we can find a generator that uses only that edge-label; otherwise, we need a generator that is an indicator spline with nonzero entry given by the product of the two edge-labels.

**Theorem 3.2.** *Let $R$ be a UFD. Let $(G, \alpha)$ be a connected edge-labeled graph with edge-labeling $\alpha\colon E \to \mathcal{I}$ having image $\{\langle \mathsf{i} \rangle, \langle \mathsf{j} \rangle\}$. Choose an ordering on $V$ as in Proposition 2.1. For every $1 < i \leq |V|$, define the spline $\mathbf{b}^i$ as follows. Choose some $v_j \in N_G(\{v_i\})$ with $j < i$, and write $\langle \mathsf{k} \rangle := \alpha(v_i v_j)$. Let*

$$G' = G - \{uv \in E \mid \alpha(uv) = \langle \mathsf{k} \rangle\},$$

*and let $C = (V', E')$ be the connected component of $G'$ containing $v_i$.*

*(a) If $V' \subseteq \{v_i, v_{i+1}, \ldots, v_{|V|}\}$, then set $\mathbf{b}^i_u = \mathsf{k}$ for all $u \in V'$ and $\mathbf{b}^i_u = 0$ for all $u \notin V'$.*

*(b) Otherwise, set $\mathbf{b}^i_{v_i} = \operatorname{lcm}(\mathsf{i}, \mathsf{j})$ and $\mathbf{b}^i_u = 0$ for all $u \neq v_i$.*

*Then $R_{G,\alpha}$ is a free $R$-module, and the set $\mathcal{B} = \{\mathbf{1}, \mathbf{b}^2, \ldots, \mathbf{b}^{|V|}\}$ is a basis for $R_{G,\alpha}$.*

**Remark 3.3.** Note that if the image of $\alpha$ is a single edge-label, then $C$ always consists of the single vertex $v_i$ and Case (b) never applies. Thus, Theorem 3.1 is a special case of Theorem 3.2.

*Proof.* We prove that $\mathcal{B}$ is an MGS for $R_{G,\alpha}$. Since $R$ is a domain, Lemma 2.6 then implies the claim.

We first show that $\mathbf{b}^i$ is a spline in $R_{G,\alpha}$ for all $i > 1$. If $\mathbf{b}^i$ was produced by Case (a), then for every edge $uw \in E$

$$\mathbf{b}^i_u - \mathbf{b}^i_w = \begin{cases} \mathsf{k} - \mathsf{k} = 0 & \text{if both } u, w \text{ are in } V', \\ 0 - 0 = 0 & \text{if neither } u, w \text{ are in } V', \\ \pm(\mathsf{k} - 0) = \pm \mathsf{k} & \text{if exactly one of } u, w \text{ are in } V'. \end{cases}$$

The first two cases satisfy the GKM condition trivially; the last one does because here the edge $uw$ was deleted from $G$ to form $G'$, so $\alpha(uw) = \langle \mathsf{k} \rangle$. If $\mathbf{b}^i$ was produced by Case (b), then for every edge $uw \in E$ the difference $\mathbf{b}^i_u - \mathbf{b}^i_w$ is either zero or a nonzero element of $I \cap J$. Thus, $\mathbf{b}^i$ satisfies the GKM condition.

Now we show that $\mathcal{B} = \{\mathbf{1}, \mathbf{b}^2, \ldots, \mathbf{b}^{|V|}\}$ is an MGS for $R_{G,\alpha}$. Let $\mathbf{p} \in R_{G,\alpha}$ be an arbitrary spline. We claim that there exist $r_2, \ldots, r_{|V|} \in R$ such that

$$\mathbf{p} = \mathbf{p}_{v_1}\mathbf{1} + r_2\mathbf{b}^2 + \cdots + r_{|V|}\mathbf{b}^{|V|}. \tag{2}$$

For the base case, first note that the spline $\mathbf{p} - \mathbf{p}_{v_1}\mathbf{1}$ satisfies $(\mathbf{p} - \mathbf{p}_{v_1}\mathbf{1})_{v_1} = 0$, so $\mathbf{p}_{v_1}\mathbf{1}$ agrees with $\mathbf{p}$ at the first vertex. Now assume that we can find coefficients $r_2, \ldots, r_m \in R$ so that the spline $\mathbf{p}_{v_1}\mathbf{1} + r_2\mathbf{b}^2 + \cdots + r_m\mathbf{b}^m$ agrees with $\mathbf{p}$ when evaluated at the first $m$ vertices.

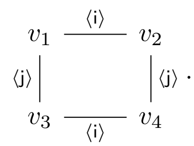
By Proposition 2.1, there is some $v_k \in N_G(\{v_{m+1}\})$ with $k < m+1$. Without loss of generality, assume $\alpha(v_{m+1}v_k) = \langle \mathsf{i} \rangle$. Then either

(i) $\mathbf{p}_{v_{m+1}} - \mathbf{p}_{v_k} \in I \cap (I \cap J)^c$ or

(ii) $\mathbf{p}_{v_{m+1}} - \mathbf{p}_{v_k} \in I \cap J$.

We show that these two cases correspond to Cases (a) and (b), respectively, in the statement of the theorem. Suppose Case (b) applies, namely there is a path between $v_{m+1}$ and one of the vertices $v_1, \ldots, v_m$ all of whose edges are labeled $J$. The GKM conditions along this path imply that $\mathbf{p}_{v_{m+1}} - \mathbf{p}_{v_k} \in J$. This forces the spline to have the form in Case (ii). Otherwise Case (a) applies and thus Case (i) is possible. In both cases, there exists $r_{m+1}$ so that $\mathbf{p}_{v_1}\mathbf{1} + r_2\mathbf{b}^2 + \cdots + r_m\mathbf{b}^m + r_{m+1}\mathbf{b}^{m+1}$ agrees with $\mathbf{p}$ when evaluated at the first $m+1$ vertices, as desired.

By induction, Equation (2) holds; thus, $\mathcal{B}$ is a set of $|V|$ generators for $R_{G,\alpha}$. Lemma 2.7 guarantees that any MGS for $R_{G,\alpha}$ has at least $|V|$ elements, so $\mathcal{B}$ is an MGS for $R_{G,\alpha}$. □

**Example 3.4.** Consider the edge-labeled graph 

. Note that we have chosen an ordering on the vertices as in Theorem 3.2 (or Proposition 2.1).

To produce $\mathbf{b}^2$, we look at all vertices that are connected to $v_2$ by paths labeled exclusively $\langle j \rangle$. This gives the set $C' = \{v_2, v_4\}$. Thus we are in Case (a), so $\mathbf{b}^2$ is zero on $\{v_1, v_3\}$ and $\mathsf{i}$ otherwise.

Similarly, to find $\mathbf{b}^3$ we get the connected component $C' = \{v_3, v_4\}$ and are again in Case (a). In this case, $\mathbf{b}^3$ is zero on $\{v_1, v_2\}$ and $\mathsf{j}$ otherwise.

However, when constructing $\mathbf{b}^4$ we find that $C' = \{v_2, v_4\}$. Thus $\mathbf{b}^4$ is $\operatorname{lcm}(\mathsf{i}, \mathsf{j})$ on $v_4$ and zero otherwise.

The set $\mathcal{B} = \{\mathbf{1}, \mathbf{b}^2, \mathbf{b}^3, \mathbf{b}^4\}$ is an MGS for $R_{G,\alpha}$ by Theorem 3.2.

**Remark 3.5.** The previous example describes the cycles that correspond to what are called *singular vertices* in the literature on splines. See Section 5 for more.

## 4. Polynomial splines on cycles

In Section 3, we produced MGSs for arbitrary connected graphs, as long as there were at most two edge-labels. In this section, we treat an arbitrary number of edge-labels, but we restrict the types of graphs and ideals under consideration.

4.1. **Degree sequences for splines.** Let $R = \Bbbk[x_1, \ldots, x_m]$ with $\Bbbk$ a field, and let $(G, \alpha)$ be an arbitrary edge-labeled graph. Recall that throughout this paper, we assume that all ideals in the image of $\alpha$ are principal (see Remark 2.8). We now add the assumption that the ideals are generated by homogeneous elements and introduce an invariant of $(G, \alpha)$ called the "degree sequence". (As described in the introduction and in Section 5, homogeneity is a very natural condition in geometric and analytic applications.)

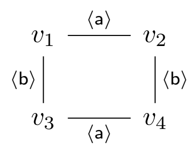
**Definition 4.1.** An MGS $\mathcal{B} = \{\mathbf{b}^1, \ldots, \mathbf{b}^n\}$ is called *homogeneous* if, for each $1 \leq i \leq n$, every nonzero entry of $\mathbf{b}^i$ is a homogeneous polynomial of the same degree, which we denote as $\deg \mathbf{b}^i$.

**Definition 4.2.** Let $\mathcal{B}$ be a homogeneous MGS. For each $j \in \mathbb{Z}_{\geq 0}$, let $d_j = \left|\{\mathbf{b} \in \mathcal{B} \mid \deg \mathbf{b} = j\}\right|$. Then the *degree sequence* of $\mathcal{B}$ is defined as $\overline{d}_{\mathcal{B}} = (d_0, d_1, d_2, d_3, \ldots)$.

**Remark 4.3.** The degree sequence only has a finite number of nonzero entries. For instance, when edge-labels are principal ideals, no generator need have larger degree than that of the product of the edge-labels. In particular, $d_m = 0$ if $m$ is greater than the sum of the degrees of the edge-labels. (See also [GTV16, Corollary 5.2].)

For the remainder of the paper, we only consider principal ideals generated by degree-two elements of the form $(x + ay)^2$ with $0 \neq a \in \Bbbk$. (See Section 5 for how this case appears in the study of classical analytic splines.) For convenience, we denote these edge-labels using a sans-serif letter; for example, we write $\mathsf{a} := (x + ay)^2$.

**Example 4.4.** Let $\Bbbk$ be a field, and let $R = \Bbbk[x, y]$. Consider the edge-labeled graph $(G, \alpha)$ given by




where $0 \neq a, b \in \Bbbk$. This is a specialization of the edge-labeled graph in Example 3.4. Theorem 3.2 asserts that

$$\mathcal{B} = \{\mathbf{1}, (0, \mathsf{a}, 0, \mathsf{a}), (0, 0, \mathsf{b}, \mathsf{b}), (0, 0, 0, \mathsf{ab})\}$$

is a homogeneous MGS for $R_{G,\alpha}$. The degree sequence of $\mathcal{B}$ is thus $\overline{d}_{\mathcal{B}} = (1, 0, 2, 0, 1)$.

We next prove that the degree sequence is an invariant of an edge-labeled graph.

**Proposition 4.5.** *Let $(G, \alpha)$ be a connected edge-labeled graph with edges labeled by principal polynomial ideals with homogeneous generators. Let $\mathcal{B}$ and $\mathcal{B}'$ be two homogeneous MGSs for $R_{G,\alpha}$ with degree sequences $\overline{d}_{\mathcal{B}}$ and $\overline{d}_{\mathcal{B}'}$, respectively. Then $\overline{d}_{\mathcal{B}} = \overline{d}_{\mathcal{B}'}$.*

*Proof.* Let $\mathcal{B} = \{\mathbf{b}^1, \ldots, \mathbf{b}^n\}$ and $\mathcal{B}' = \{\mathbf{b}^{1'}, \ldots, \mathbf{b}^{n'}\}$ be two homogeneous MGSs for $R_{G,\alpha}$ with degree sequences $\overline{d}_{\mathcal{B}} = (d_0, d_1, d_2, \ldots)$ and $\overline{d}_{\mathcal{B}'} = (d'_0, d'_1, d'_2, \ldots)$. (If needed, add terminal zeros so both sequences have the same length.) We show that $d_r = d'_r$ for each $r$ by induction on the index $r$ and prove as our base case that $d_0 = d'_0$. The degree-zero splines in $R_{G,\alpha}$ generate a $\Bbbk$-vector space. The degree-zero splines in $\mathcal{B}$ form an MGS for the degree-zero splines in $R_{G,\alpha}$, and likewise for $\mathcal{B}'$. Since MGSs in vector spaces are bases, and in particular have the same number of elements, the base case of our induction holds. Now assume that $d_0 = d'_0, d_1 = d'_1, \ldots, d_{r-1} = d'_{r-1}$.

Given $\mathbf{b}^{i'} \in \mathcal{B}'$, we can write $\mathbf{b}^{i'} = k_{i,1}\mathbf{b}^1 + k_{i,2}\mathbf{b}^2 + \cdots + k_{i,n}\mathbf{b}^n$ for some coefficients $k_{i,1}, \ldots, k_{i,n} \in R$. Note that

$$\sum_{\substack{\mathbf{b}^j \in \mathcal{B} \\ \deg \mathbf{b}^{i'} < \deg \mathbf{b}^j}} k_{i,j}\mathbf{b}^j = 0,$$

so we may assume $k_{i,j} = 0$ for all $\deg \mathbf{b}^{i'} < \deg \mathbf{b}^j$. It follows that

$$\{\mathbf{b}^{i'} \in \mathcal{B}' \mid \deg \mathbf{b}^{i'} \leq r\} \subseteq \operatorname{span}(\{\mathbf{b}^i \in \mathcal{B} \mid \deg \mathbf{b}^i \leq r\}).$$

A symmetric argument shows that

$$\{\mathbf{b}^{i} \in \mathcal{B} \mid \deg \mathbf{b}^{i} \leq r\} \subseteq \operatorname{span}(\{\mathbf{b}^{i'} \in \mathcal{B}' \mid \deg \mathbf{b}^{i'} \leq r\}).$$

This contradicts minimality of $\mathcal{B}$ or $\mathcal{B}'$ unless $\sum_{i=0}^r d_i = \sum_{i=0}^r d'_i$. By the inductive hypothesis, this implies $d_r = d'_r$. □

4.2. **Linear algebraic background.** We use the following two linear-algebraic results about the $\Bbbk$-vector space of polynomials.

**Lemma 4.6.** *Let $a, b, c, d, D \in \Bbbk$, with $a, b, c$ distinct. Then we can find unique $A, B, C \in \Bbbk$ such that*

$$A\mathsf{a} + B\mathsf{b} + C\mathsf{c} = D\mathsf{d}. \tag{3}$$

*Proof.* We rewrite Equation (3), collecting coefficients, as

$$(A + B + C)x^2 + (2aA + 2bB + 2cC)xy + (a^2A + b^2C + c^2C)y^2 = Dx^2 + 2Ddxy + Dd^2y^2.$$

Solving for $A, B, C$ amounts to solving the following system of linear equations:

$$A + B + C = D$$
$$2aA + 2bB + 2cC = 2dD$$
$$a^2A + b^2B + c^2C = d^2D.$$

The coefficient matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 2a & 2b & 2c \\ a^2 & b^2 & c^2 \end{bmatrix}$$

can be reduced to the identity matrix via elementary row operations. (Some of the row operations require division, but we avoid division by zero because $a, b, c$ are distinct.) This implies that the coefficient matrix is invertible; thus, there exists a unique solution to the system of equations. □

**Lemma 4.7.** *Let $a, b, c, C_1, C_2 \in \Bbbk$, with $a$ and $b$ distinct. There exist unique $A_1, A_2, B_1, B_2 \in \Bbbk$ such that*

$$(A_1x + A_2y)\mathsf{a} + (B_1x + B_2y)\mathsf{b} = (C_1x + C_2y)\mathsf{c}. \tag{4}$$

*Proof.* Expanding Equation (4) and equating coefficients of like terms leads to the following system of linear equations:

$$\begin{aligned} A_1 + B_1 &= C_1 \\ 2aA_1 + A_2 + 2bB_1 + B_2 &= 2cC_1 + C_2 \\ a^2A_1 + 2aA_2 + b^2B_1 + 2bB_2 &= c^2C_1 + 2cC_2 \\ a^2A_2 + b^2B_2 &= c^2C_2. \end{aligned}$$

The coefficient matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2a & 1 & 2b & 1 \\ a^2 & 2a & b^2 & 2b \\ 0 & a^2 & 0 & b^2 \end{bmatrix}$$

can be reduced to the identity matrix via elementary row operations. (Some of the row operations require division, but we avoid division by zero because $a \neq b$.) This implies that the coefficient matrix is invertible; thus, there exists a unique solution to the system of equations. □

4.3. **Constructions to reduce graphs.** A product module has a collection of forgetful maps to different factors in the module. Suppose $(G', \alpha')$ is an edge-labeled graph obtained from another edge-labeled graph $(G, \alpha)$ by adding a single vertex $v$ together with some labeled edges from $v$ to vertices in $G$. Then we can use the forgetful map to relate the splines on $(G', \alpha')$ to those on $(G, \alpha)$.

This is what we do in the next result. We then specialize to the case of cycles in Corollary 4.12. We note that Lemma 4.8 below also applies to general edge-labelings $\alpha$, which we will restrict in more ways throughout this section.

**Lemma 4.8.** *Suppose that $(G, \alpha)$ and $(G', \alpha')$ are edge-labeled graphs with vertices $V' = V \cup \{v\}$, edges*

$$E' = E \cup \{vu \mid u \in U\}$$

*where $U \subseteq V$ is nonempty, and edge-labeling $\alpha'|_E = \alpha$. Then the projection map $\bigoplus_{u \in V'} R \to \bigoplus_{u \in V} R$ induces an $R$-module homomorphism $\varphi\colon R_{G',\alpha'} \to R_{G,\alpha}$, and*

$$R_{G',\alpha'} \cong \ker\varphi \oplus \operatorname{im}\varphi.$$

*Moreover, suppose that every pair $u, u' \in U$ is connected by a path of edges in $E$ all labeled $I$, and suppose that $\alpha'(vu) = I$ for all $u \in U$. Then*

$$R_{G',\alpha'} \cong I \oplus R_{G,\alpha}.$$

*Proof.* We first show that restricting a spline $\mathbf{p}$ in $R_{G',\alpha'}$ to the set of vertices $V$ produces a spline in $R_{G,\alpha}$. Indeed, for each edge $uu' \in E$ we have

$$\mathbf{p}_u - \mathbf{p}_{u'} \in \alpha'(uu') = \alpha(uu').$$

Thus the projection map induces an $R$-module homomorphism $\varphi\colon R_{G',\alpha'} \to R_{G,\alpha}$. We conclude that

$$R_{G',\alpha'} \cong \ker\varphi \oplus \operatorname{im}\varphi.$$

Now we consider the special case where every pair $u, u' \in U$ is connected by a path of edges in $E$ all labeled $I$, and $\alpha'(vu) = I$ for all $u \in U$. Note that $\ker\varphi$ consists of all splines in $R_{G',\alpha'}$ that are zero at all of $V$. Consider a spline in $\ker\varphi$. If at least one edge incident to $v$ is labeled $I$, then the vertex $v$ must be labeled by an element of $I$ by the GKM condition; if all edges incident to $v$ are labeled $I$, then every element of $I$ works. Thus $\ker\varphi \cong I$.

Given a spline $\mathbf{q} \in R_{G,\alpha}$, we define $\mathbf{p} \in R_{G',\alpha'}$ such that $\varphi(\mathbf{p}) = \mathbf{q}$ according to the rule $\mathbf{p}_u = \mathbf{q}_u$ for all $u \in V$ and $\mathbf{p}_v = \mathbf{q}_{u'}$ for some $u' \in U$. The GKM condition implies that $\mathbf{q}_u - \mathbf{q}_{u'} \in I$ for any $u \in U$, since $u$ and $u'$ are connected by a path of edges labeled $I$ by hypothesis. Thus, we have $\mathbf{p}_u - \mathbf{p}_v = \mathbf{q}_u - \mathbf{q}_{u'} \in I$ for all $u \in U$. By inspection of the GKM conditions, we conclude $\mathbf{p} \in R_{G',\alpha'}$ and thus $\operatorname{im}\varphi \cong R_{G,\alpha}$. □

We can (and will) use Lemma 4.8 to eliminate those vertices whose incident edges all have the same label. This leads us to the following definition.

**Definition 4.9.** An edge-labeled graph is called *reduced* if no two edges that are incident to the same vertex have the same edge-label.

We note that the edge-labels in a reduced cycle have to be at least somewhat evenly distributed, in the following sense.

**Lemma 4.10.** *Suppose $(G, \alpha)$ is a reduced edge-labeled graph. Moreover, suppose $(G, \alpha)$ contains $n$ distinct vertices $v_0, v_1, v_2, \ldots, v_n = v_0$ that form a cycle with three or more distinct edge labels. Then there is at least one sequence of three successive distinct edge-labels on the edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$.*

*Proof.* Read clockwise around the cycle starting at an arbitrary edge, and suppose the first two edges are labeled $I$ and $J$. If a sequence of three successive edge-labels does not contain three distinct edge-labels, then it must alternate between two of them since the graph is reduced. An edge-label that is neither $I$ nor $J$ appears somewhere on the graph by hypothesis of at least three distinct edge-labels. Look at the first occurrence of this edge-label in the sequence; the two edges preceding it have labels from the set $\{I, J\}$, without repetition. This proves the claim. □

In the next lemma, we refine Lemma 4.8 to keep track of MGSs. Note that it assumes edge-labelings by principal ideals in UFDs, not necessarily polynomials.

**Lemma 4.11.** *Let $(G, \alpha)$ and $(G', \alpha')$ be defined as in Lemma 4.8, with the condition that every pair $u, u' \in U$ is connected by a path of edges in $E$ all labeled $\langle \mathsf{i} \rangle$ and $\alpha'(vu) = \langle \mathsf{i} \rangle$ for all $u \in U$. Let $|V| = n$ and fix some $u' \in U$. If $\mathcal{B} = \{\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^n\}$ is an MGS for $R_{G,\alpha}$, then $\mathcal{B}' = \{\mathbf{b}^{1'}, \mathbf{b}^{2'}, \ldots, \mathbf{b}^{n'}, \mathbf{b}^{n+1}\}$, where*

$$\mathbf{b}^{i\,\prime}_u = \begin{cases} \mathbf{b}^i_u & \text{if } u \in V, \\ \mathbf{b}^i_{u'} & \text{if } u = v, \end{cases}$$

*and*

$$\mathbf{b}^{n+1}_u = \begin{cases} 0 & \text{if } u \in V, \\ \mathsf{i} & \text{if } u = v, \end{cases}$$

*is an MGS for $R_{G',\alpha'}$.*

*Proof.* It is clear that $\mathbf{b}^{n+1}$ is a spline in $R_{G',\alpha'}$. If $\varphi$ is the map from Lemma 4.8, then $\varphi(\mathbf{b}^{i'}) = \mathbf{b}^i$ for all $1 \le i \le n$ by construction, so (the last paragraph of the proof of) Lemma 4.8 implies that $\mathbf{b}^{i'} \in R_{G',\alpha'}$ again for all $1 \le i \le n$.

We now show that $\mathcal{B}'$ is a generating set for $R_{G',\alpha'}$. Each spline $\mathbf{p} \in R_{G',\alpha'}$ satisfies

$$\mathbf{p}_v = \mathbf{p}_{u'} + k\mathsf{i} \tag{5}$$

for some $k \in R$ by the GKM condition for the edge $vu'$. The spline $\varphi(\mathbf{p})$ is an element of $R_{G,\alpha}$ with

$$\varphi(\mathbf{p})_u = \mathbf{p}_u \qquad \text{for all } u \in V. \tag{6}$$

Because $\mathcal{B}$ is an MGS for $R_{G,\alpha}$, we can write $\varphi(\mathbf{p})$ as a linear combination

$$\varphi(\mathbf{p}) = r_1\mathbf{b}^1 + r_2\mathbf{b}^2 + \cdots + r_n\mathbf{b}^n, \tag{7}$$

where each $r_i \in R$. For all $u \in V$, we have

$$\begin{aligned}
\mathbf{p}_u &= \varphi(\mathbf{p})_u && \text{by Equation (6)},\\
&= r_1\mathbf{b}^1_u + r_2\mathbf{b}^2_u + \cdots + r_n\mathbf{b}^n_u + k\cdot 0 && \text{by Equation (7)},\\
&= r_1\mathbf{b}^{1'}_u + r_2\mathbf{b}^{2'}_u + \cdots + r_n\mathbf{b}^{n'}_u + k\mathbf{b}^{n+1}_u && \text{by the definition of } \mathcal{B}'.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathbf{p}_v &= \mathbf{p}_{u'} + k\mathsf{i} && \text{by Equation (5)},\\
&= r_1\mathbf{b}^1_{u'} + r_2\mathbf{b}^2_{u'} + \cdots + r_n\mathbf{b}^n_{u'} + k\mathsf{i} && \text{by the previous argument},\\
&= r_1\mathbf{b}^{1'}_v + r_2\mathbf{b}^{2'}_v + \cdots + r_n\mathbf{b}^{n'}_v + k\mathbf{b}^{n+1}_v && \text{by the definition of } \mathcal{B}'.
\end{aligned}$$

We have obtained the equation

$$\mathbf{p} = r_1\mathbf{b}^{1'} + r_2\mathbf{b}^{2'} + \cdots + r_n\mathbf{b}^{n'} + k\mathbf{b}^{n+1},$$

which assures that $\mathcal{B}'$ is a generating set for $R_{G',\alpha'}$.

Moreover, the set $\mathcal{B}'$ is an MGS by Lemma 2.7 because it consists of $n+1$ elements and $G'$ is a graph with $n+1$ vertices. □

We now apply the ideas in the previous lemma to the case of cycles, which is the special case on which we focus.

**Corollary 4.12.** *Let $(C_n, \alpha_n)$ be an edge-labeled $n$-cycle. Create an edge-labeled $(n+1)$-cycle $(C_{n+1}, \alpha_{n+1})$ from $(C_n, \alpha_n)$ by inserting a vertex $v_{n+1}$ into the edge $v_nv_1$ with both new edges $v_nv_{n+1}$ and $v_{n+1}v_1$ labeled the same as $v_nv_1$ was. Then*

$$R_{C_{n+1},\alpha_{n+1}} \cong \alpha_n(v_nv_1) \oplus R_{C_n,\alpha_n}.$$

*Moreover, suppose $(C_n, \alpha_n)$ has edges labeled with principal ideals generated by homogeneous polynomials, that $(C_n, \alpha_n)$ has MGS $\mathcal{B}$, and that the generator of the edge-label $\alpha_n(v_nv_1)$ is a homogeneous polynomial of degree $e$. Then $(C_{n+1}, \alpha_{n+1})$ has an MGS $\mathcal{B}'$ that*

1. *extends $\mathcal{B}$ in the sense that if $\varphi$ is the map from Lemma 4.8 then $\varphi(\mathcal{B}') \supseteq \mathcal{B}$,*
2. *has exactly one more generator than $\mathcal{B}'$ and the degree of this additional generator is $e$, and*
3. *is minimal in the sense that if $\mathcal{B}''$ is any other generating set that extends $\mathcal{B}$ then $\mathcal{B}''$ has at least one more element of degree $e$ than $\mathcal{B}$ (and possibly other additional elements of other degrees).*

*In particular, the degree sequence of $\mathcal{B}'$ satisfies*

$$\overline{d}_{\mathcal{B}'} = \overline{d}_{\mathcal{B}} + \left(0^{e-1}, 1, 0^{m-e}\right).$$

*Proof.* Taking $(G, \alpha)$ to be $(C_n, \alpha_n)$ and $(G', \alpha')$ to be the (non-cyclic) edge-labeled graph formed from $(C_n, \alpha_n)$ by adding a new vertex $v_{n+1}$ and new edges $v_nv_{n+1}$ and $v_{n+1}v_1$ labeled the same as $v_nv_1$, we can apply Lemma 4.8 to conclude

$$R_{G',\alpha'} \cong \alpha_n(v_nv_1) \oplus R_{C_n,\alpha_n}.$$

Note that $R_{G',\alpha'} \subseteq R_{C_{n+1},\alpha_{n+1}}$ because $(G', \alpha')$ consists of the edge-labeled graph $(C_{n+1}, \alpha_{n+1})$ together with precisely one additional edge. The three edges $v_nv_1$, $v_nv_{n+1}$, and $v_{n+1}v_1$ in $(G', \alpha')$ all have the same label, so every spline in $R_{C_{n+1},\alpha_{n+1}}$ satisfies the GKM conditions on $(G', \alpha')$. Thus $R_{G',\alpha'} \cong R_{C_{n+1},\alpha_{n+1}}$. In particular, if $\alpha_n(v_nv_1)$ is a principal ideal generated by a homogeneous polynomial of degree $e$ and if $\mathcal{B}$ is an MGS of $(C_n, \alpha_n)$, then Lemma 4.11 constructs an MGS for

$(C_{n+1}, \alpha_{n+1})$ that satisfies Conditions (1) and (2) of our claim. The explicit description of the degree sequence $\overline{d}_{\mathcal{B}'}$ of $(C_{n+1}, \alpha_{n+1})$ follows from the definition of degree sequence and from Conditions (1) and (2). The minimality in Condition (3) follows from the direct sum decomposition

$$R_{C_{n+1},\alpha_{n+1}} \cong \alpha_n(v_n v_1) \oplus R_{C_n,\alpha_n}$$

since any generating set $\mathcal{B}''$ that extends $\mathcal{B}$ needs at least one additional element of degree $e$ to generate $\alpha_{n+1}$. □

4.4. **Producing an MGS for polynomial edge-labeled cycles.** We now construct an algorithm that produces a homogeneous MGS for cycles whose edges are labeled by principal polynomial ideals with generator of the form $\mathsf{a} := (x + ay)^2$ for $a \neq 0$. Part of our proof proceeds by induction; the following lemma proves the base case of a triangle.

**Lemma 4.13.** *Let $(G, \alpha)$ be a 3-cycle with edge-labeling $\alpha\colon E \to \mathcal{I}$ having $\alpha(v_1v_2) = \langle \mathsf{a} \rangle$, $\alpha(v_2v_3) = \langle \mathsf{b} \rangle$, and $\alpha(v_3v_1) = \langle \mathsf{c} \rangle$ so that $a, b, c$ are all distinct. Let $f_1, f_2, g_1, g_2 \in \Bbbk[x, y]$ denote the homogeneous degree-one polynomials with $x\mathsf{a} = f_1\mathsf{b} + g_1\mathsf{c}$ and $y\mathsf{a} = f_2\mathsf{b} + g_2\mathsf{c}$ that are guaranteed by Lemma 4.7. Then the set*

$$\mathcal{B} = \{\mathbf{1}, \mathbf{b}^2, \mathbf{b}^3\} = \{\mathbf{1}, (0, x\mathsf{a}, g_1\mathsf{c}), (0, y\mathsf{a}, g_2\mathsf{c})\}$$

*is a homogeneous basis for $R_{G,\alpha}$.*

*Proof.* We will prove that $\mathcal{B}$ is a homogeneous MGS that is also free. Note that $(0, x\mathsf{a}, g_1\mathsf{c})$ is a spline: the GKM condition on the edges labeled $\mathsf{a}$ and $\mathsf{c}$ are trivially satisfied, and the condition on the edge labeled $\mathsf{b}$ is satisfied because $x\mathsf{a} = f_1\mathsf{b} + g_1\mathsf{c}$. The same argument shows that $(0, y\mathsf{a}, g_2\mathsf{c})$ is a spline. Moreover, the GKM condition for spline $\mathbf{b}^2$ on edge $v_2v_3$ implies $g_1 \neq x$ since $\mathsf{b} = (x + by)^2$ cannot divide the polynomial

$$x\mathsf{a} - x\mathsf{c} = xy(2ax - 2cx + a^2y - c^2y)$$

by direct computation (or by noting that both $\mathsf{b}$ and the right-hand side of the displayed equation are factored into irreducibles, and that the polynomial ring $\Bbbk[x, y]$ is a UFD). A similar argument shows $g_2 \neq y$.

Now we demonstrate that $\mathcal{B}$ generates the spline $(0, 0, \mathsf{bc})$. We have

$$yg_1\mathsf{c} = xy\mathsf{a} - yf_1\mathsf{b} \quad \text{and} \quad xg_2\mathsf{c} = xy\mathsf{a} - xf_2\mathsf{b}.$$

Subtracting, we obtain the equality

$$(yg_1 - xg_2)\mathsf{c} = (xf_2 - yf_1)\mathsf{b}.$$

If $(yg_1 - xg_2)\mathsf{c} = 0$ then the degree-two factor $yg_1 - xg_2$ is identically zero, and so $g_1 = rx$ and $g_2 = ry$ for some scalar $r$. Plugging this back into the equation $yg_1\mathsf{c} = xy\mathsf{a} - yf_1\mathsf{b}$ and then rearranging, we obtain $x(\mathsf{a} - r\mathsf{c}) = f_1\mathsf{b}$ and similarly $y(\mathsf{a} - r\mathsf{c}) = f_2\mathsf{b}$. Multiplying these two equations by $y$ and $x$ respectively, we obtain $yf_1 = xf_2$. Analyzing degree constraints once more, we conclude $f_1 = sx$ and $f_2 = sy$ for some scalar $s$. Plugging this back into the equation $x\mathsf{a} = f_1\mathsf{b} + g_1\mathsf{c}$, we have $x\mathsf{a} = sx\mathsf{b} + rx\mathsf{c}$. In particular $\mathsf{a} = s\mathsf{b} + r\mathsf{c}$, which contradicts the linear independence of $\mathsf{a}$, $\mathsf{b}$, and $\mathsf{c}$ over $\Bbbk$ proved in Lemma 4.6.

Thus $(yg_1 - xg_2)\mathsf{c}$ is a homogeneous degree-four polynomial that is divisible by both $\mathsf{b}$ and $\mathsf{c}$. It must be a scalar multiple of $\mathsf{bc}$ because $\mathsf{b}$ and $\mathsf{c}$ have no irreducible factors in common. Consequently, the spline

$$\mathbf{q} := y\mathbf{b}^2 - x\mathbf{b}^3 = (0, 0, (yg_1 - xg_2)\mathsf{c})$$

is a nonzero scalar multiple of $(0, 0, \mathsf{bc})$.

Now we show the generators are actually free, namely that if

$$p_1\mathbf{1} + p_2\mathbf{b}^2 - p_3\mathbf{b}^3 = (0, 0, 0)$$

then $p_i = 0$ for all $i \in \{1, 2, 3\}$. The first coordinate shows that $p_1 = 0$ since on the left-hand side we have

$$\left(p_1\mathbf{1} + p_2\mathbf{b}^2 - p_3\mathbf{b}^3\right)_{v_1} = p_1\mathbf{1}_{v_1} + 0 - 0 = p_1.$$

Using $p_1 = 0$ and the explicit equations for $\mathbf{b}^1, \mathbf{b}^2$, we obtain

$$(0, p_2 x\mathsf{a} - p_3 y\mathsf{a}, p_2 g_1 \mathsf{c} - p_3 g_2 \mathsf{c}) = (0, 0, 0).$$

Since $p_2 x\mathsf{a} - p_3 y\mathsf{a} = 0$ in a UFD, we conclude as above that $x$ divides $p_3$ and $y$ divides $p_2$. Write $p_3 = p_3' x$ and $p_2 = p_2' y$. Then we have

$$p_2' y x\mathsf{a} - p_3' x y\mathsf{a} = (p_2' - p_3') x y\mathsf{a} = 0$$

and so $p_2' = p_3'$. Now examining the last coordinate, we see

$$p_2' y g_1 \mathsf{c} - p_3' x g_2 \mathsf{c} = (p_2')\left((y g_1 - x g_2)\mathsf{c}\right) = 0.$$

We just proved that $(yg_1 - xg_2)\mathsf{c}$ is a nonzero scalar multiple of $\mathsf{bc}$, so this entry is zero if and only if $p_2' = 0$. Hence all $p_i$ are zero, as desired.

Finally we show that $\mathcal{B}$ generates an arbitrary spline $\mathbf{p} \in R_{G,\alpha}$. We have

$$\mathbf{p} - \mathbf{p}_{v_1}\mathbf{1} = (0, \mathbf{p}_{v_2} - \mathbf{p}_{v_1}, \mathbf{p}_{v_3} - \mathbf{p}_{v_1}).$$

By the GKM conditions on edges $v_1v_2$ and $v_3v_1$, we have $\mathbf{p}_{v_2} - \mathbf{p}_{v_1} = k\mathsf{a}$ and $\mathbf{p}_{v_3} - \mathbf{p}_{v_1} = \ell\mathsf{c}$ for some $k, \ell \in \Bbbk[x, y]$. The GKM condition on edge $v_2v_3$ gives the equation

$$(\mathbf{p}_{v_2} - \mathbf{p}_{v_1}) - (\mathbf{p}_{v_3} - \mathbf{p}_{v_1}) = k\mathsf{a} - \ell\mathsf{c} = \ell'\mathsf{b}$$

for some $\ell' \in \Bbbk[x, y]$. Lemma 4.6 showed that $\mathsf{a}$, $\mathsf{b}$, and $\mathsf{c}$ are linearly independent over the base field $\Bbbk$, so the only scalar solution to $k\mathsf{a} - \ell\mathsf{c} - \ell'\mathsf{b} = 0$ is $k = \ell = \ell' = 0$. Thus $k, \ell, \ell'$ are polynomials without constant terms. Assume that $h_1, h_2 \in \Bbbk[x, y]$ satisfy $k = h_1 x + h_2 y$. We have

$$\mathbf{p}_{v_2} - \mathbf{p}_{v_1} = (h_1 x + h_2 y)\mathsf{a}$$

and

$$\mathbf{p} - \mathbf{p}_{v_1}\mathbf{1} - h_1\mathbf{b}^2 - h_2\mathbf{b}^3 = (0, 0, \mathbf{p}_{v_3} - \mathbf{p}_{v_1} - h_1 g_1 \mathsf{c} - h_2 g_2 \mathsf{c}).$$

The nonzero entry in this spline must be a multiple of both $\mathsf{b}$ and $\mathsf{c}$ by the GKM conditions on edges $v_2v_3$ and $v_3v_1$, respectively. Hence

$$\mathbf{p} - \mathbf{p}_{v_1}\mathbf{1} - h_1\mathbf{b}^2 - h_2\mathbf{b}^3 = t\mathbf{q}$$

for some $t \in \Bbbk[x, y]$ because we showed above that $\mathbf{q}$ is a scalar multiple of $(0, 0, \mathsf{bc})$. We conclude that $\mathcal{B}$ generates $R_{G,\alpha}$. Lemma 2.7 asserts that $\mathcal{B}$ is an MGS as desired. □

The heart of the proof of Theorem 4.15, our main theorem about cycles, is the following lemma. After proving the lemma, Theorem 4.15 will follow easily by applying the reduction lemmas from Section 4.3.

**Lemma 4.14.** *Let $(G, \alpha)$ be an edge-labeled $n$-cycle containing a sequence of three successive distinct edge-labels. Order the vertices $v_0, v_1, v_2, \ldots, v_n = v_0$ of $(G, \alpha)$ clockwise around the cycle such that $\alpha(v_{i-1}v_i) = \langle \mathsf{a}_i \rangle$ and $a_{n-1}, a_n$, and $a_1$ are all distinct.*

*We give an explicit homogeneous MGS $\mathcal{B} = \{\mathbf{1}, \mathbf{b}^2, \ldots, \mathbf{b}^n\}$ for $R_{G,\alpha}$ as follows. For every $1 < i \leq n-2$, let $a_{i,n-1}, a_{i,n}, a_{i,1} \in \Bbbk$ be the base field elements with $\mathsf{a}_i = a_{i,n-1}\mathsf{a}_{n-1} + a_{i,n}\mathsf{a}_n + a_{i,1}\mathsf{a}_1$ that are guaranteed by Lemma 4.6, and define $\mathbf{b}^i$ by*

$$\mathbf{b}^i_{v_j} = \begin{cases} 0 & \text{if } j < i, \\ \mathsf{a}_i & \text{if } i \leq j \leq n-2, \\ \mathsf{a}_i - a_{i,n-1}\mathsf{a}_{n-1} & \text{if } j = n-1, \\ \mathsf{a}_i - a_{i,n-1}\mathsf{a}_{n-1} - a_{i,n}\mathsf{a}_n & \text{if } j = n. \end{cases}$$

*As in Lemma 4.13, let $f_1, f_2, g_1, g_2 \in \Bbbk[x, y]$ denote the homogeneous degree-one polynomials with $x\mathsf{a}_{n-1} = f_1\mathsf{a}_n + g_1\mathsf{a}_1$ and $y\mathsf{a}_{n-1} = f_2\mathsf{a}_n + g_2\mathsf{a}_1$ that are guaranteed by Lemma 4.7. Define $\mathbf{b}^{n-1}$ by*

$$\mathbf{b}^{n-1}_{v_j} = \begin{cases} 0 & \text{if } j \leq n-2, \\ x\mathsf{a}_{n-1} & \text{if } j = n-1, \\ g_1\mathsf{a}_1 & \text{if } j = n, \end{cases}$$

*and* $\mathbf{b}^n$ *by*

$$\mathbf{b}^n_{v_j} = \begin{cases} 0 & \text{if } j \leq n-2, \\ y\mathsf{a}_{n-1} & \text{if } j = n-1, \\ g_2\mathsf{a}_1 & \text{if } j = n. \end{cases}$$

*Then* $\mathcal{B} = \{\mathbf{1}, \mathbf{b}^2, \ldots, \mathbf{b}^n\}$ *is a homogeneous basis for* $R_{G,\alpha}$ *as a free module over the polynomial ring. Consequently, the degree sequence of* $(G,\alpha)$ *is* $(1, 0, n-3, 2)$*.*

*Proof.* We will check that $\mathcal{B}$ is a homogeneous MGS and that it is free, whence we will conclude that it is a homogeneous basis for the free module $R_{G,\alpha}$.

We first check that $\mathbf{b}^i$ is a spline in $R_{G,\alpha}$ for all $i > 1$. For the $\mathbf{b}^i$ with $1 < i \leq n-2$ this is clear by the definition of $\mathbf{b}^i$, and for $\mathbf{b}^{n-1}$ (respectively $\mathbf{b}^n$) this follows from the GKM condition together with rewriting the defining equation as $x\mathsf{a}_{n-1} - g_1\mathsf{a}_1 = f_1\mathsf{a}_n$ (respectively as $y\mathsf{a}_{n-1} - g_2\mathsf{a}_1 = f_2\mathsf{a}_n$).

Now we show that $\mathcal{B}$ generates an arbitrary spline $\mathbf{p} \in R_{G,\alpha}$. We claim that there exist $r_1, r_2, \ldots, r_n \in \Bbbk[x,y]$ such that

$$\mathbf{p} = r_1\mathbf{1} + r_2\mathbf{b}^2 + \cdots + r_n\mathbf{b}^n. \tag{8}$$

For this, it is sufficient to prove that for all $1 \leq m \leq n$, we can find coefficients $r_1, r_2, \ldots, r_m \in \Bbbk[x,y]$ such that the spline $r_1\mathbf{1} + r_2\mathbf{b}^2 + \cdots + r_m\mathbf{b}^m$ agrees with $\mathbf{p}$ when evaluated at the first $m$ vertices. We will use induction up to $n-2$, then deal with $\mathbf{b}^{n-1}$ and $\mathbf{b}^n$ separately. For the base case, note that the spline $\mathbf{p} - \mathbf{p}_{v_1}\mathbf{1}$ has $(\mathbf{p} - \mathbf{p}_{v_1}\mathbf{1})_{v_1} = 0$. The inductive hypothesis asserts that we can find coefficients $r_1, r_2, \ldots, r_m \in \Bbbk[x,y]$ with $m < n-2$ so that the spline $r_1\mathbf{1} + r_2\mathbf{b}^2 + \cdots + r_m\mathbf{b}^m$ agrees with $\mathbf{p}$ when evaluated at the first $m$ vertices. In other words, assume we have found $r_1, r_2, \ldots, r_m \in \Bbbk[x,y]$ with $m < n-2$ such that

$$(\mathbf{p} - r_1\mathbf{1} - r_2\mathbf{b}^2 - \cdots - r_m\mathbf{b}^m)_{v_j} = 0$$

for all $j \leq m$. Thus by the GKM condition on edge $v_m v_{m+1}$, there exists $r_{m+1} \in \Bbbk[x,y]$ such that

$$(\mathbf{p} - r_1\mathbf{1} - r_2\mathbf{b}^2 - \cdots - r_m\mathbf{b}^m)_{v_{m+1}} = r_{m+1}\mathsf{a}_{m+1}.$$

Hence the spline $r_1\mathbf{1} + r_2\mathbf{b}^2 + \cdots + r_m\mathbf{b}^m + r_{m+1}\mathbf{b}^{m+1}$ agrees with $\mathbf{p}$ when evaluated at the first $m+1$ vertices, as desired. By induction, we have produced $r_1, r_2, \ldots, r_{n-2} \in \Bbbk[x,y]$ such that $r_1\mathbf{1} + r_2\mathbf{b}^2 + \cdots + r_{n-2}\mathbf{b}^{n-2}$ agrees with $\mathbf{p}$ when evaluated at the first $n-2$ vertices.

To conclude the proof, we essentially follow the same argument as in the proof of Lemma 4.13. Indeed, suppose $(T, \alpha')$ is the edge-labeled 3-cycle with vertices $v_1, v_{n-1}, v_n$, and with edge-labeling given by $\alpha'(v_1v_{n-1}) = \langle \mathsf{a}_{n-1} \rangle$, $\alpha'(v_{n-1}v_n) = \langle \mathsf{a}_n \rangle$, and $\alpha'(v_nv_1) = \langle \mathsf{a}_1 \rangle$. Let $\mathcal{G}$ be the subset of $R_{G,\alpha}$ in which all vertices $v_1, v_2, \ldots, v_{n-2}$ are labeled zero. Note that $\mathcal{G}$ is isomorphic to the subset of $R_{T,\alpha'}$ in which vertex $v_1$ is labeled zero, via the map $\mathcal{G} \to R_{T,\alpha'}$ that erases the initial $n-1$ zeros from each spline $\mathbf{p} \in \mathcal{G}$. Thus inserting $n-1$ leading zeros into the nontrivial generators from Lemma 4.13 gives generators for $\mathcal{G}$.

It follows that $\mathcal{B}$ generates $R_{G,\alpha}$. Indeed, we first proved that for any spline $\mathbf{p} \in R_{G,\alpha}$ we can find a *unique* linear combination of the splines $\{\mathbf{1}, \mathbf{b}^1, \ldots, \mathbf{b}^{n-2}\}$ so that $\mathbf{p} - r_0\mathbf{1} - \sum_{i=1}^{n-2} r_i\mathbf{b}^i$ is zero when evaluated at the first $n-2$ vertices. Lemma 4.13 then proved that if a spline in $R_{G,\alpha}$ is zero at the first $n-2$ vertices, it is *uniquely* generated by $\{\mathbf{b}^{n-1}, \mathbf{b}^n\}$. The generating set $\mathcal{B}$ is thus minimal and a free set of generators for the module of splines $R_{G,\alpha}$ over the polynomial ring.

Finally, the statement on the degree sequence follows because $\mathbf{1}$ is a degree-zero spline, $\mathbf{b}^i$ is a degree-two spline for all $1 < i \leq n-2$, and $\mathbf{b}^{n-1}$ and $\mathbf{b}^n$ are both degree-three splines. □

**Theorem 4.15.** *Let* $(C_n, \alpha_n)$ *be an* $n$*-cycle with three or more distinct (not necessarily successive) edge-labels. The following algorithm constructs a homogeneous MGS* $\mathcal{B}_n$ *for* $R_{C_n,\alpha_n}$*:*

1. *Let* $C_{n-k}$ *be the reduced cycle with edge-labeling* $\alpha_{n-k}$ *obtained from* $C_n$ *by eliminating vertices whose incident edges have the same label.*
2. *Let* $\mathcal{B}_{n-k}$ *be the homogeneous MGS for* $R_{C_{n-k},\alpha_{n-k}}$ *from Lemma 4.14.*
3. *Create* $\mathcal{B}_n$ *from* $\mathcal{B}_{n-k}$ *by successively reinserting vertices on repeated edges according to Corollary 4.12.*

*Proof.* The reduced cycle $C_{n-k}$ is guaranteed by Corollary 4.12. Three successive distinct edge-labels are guaranteed to exist in a reduced cycle by Lemma 4.10. Thus we may apply Lemma 4.14 to obtain the homogeneous MGS $\mathcal{B}_{n-k}$ for $R_{C_{n-k},\alpha_{n-k}}$. Reinserting each repeated edge according to Corollary 4.12 (with explicit formula given in Lemma 4.11) gives a generating set for $C_n$. Because it has the same number of elements as vertices, the final output $\mathcal{B}_n$ is an MGS for $C_n$ per Lemma 2.7. □

Example 4.17 shows an example of how to use the algorithm in Theorem 4.15 to produce a homogeneous MGS. First, we give the following corollary, which classifies degree sequences for all splines on cycles whose edge-labels are principal ideals generated by homogeneous degree-two polynomials in $\Bbbk[x, y]$.

**Corollary 4.16.** *Let $G = (V, E)$ be an $n$-cycle and let $\mathcal{I}$ be the set of principal ideals of $R = \Bbbk[x, y]$ of the form $\langle (x + ky)^2 \rangle$, where $k \in \Bbbk$. Let $\alpha\colon E \to \mathcal{I}$ be an edge-labeling of $G$. Then the following hold:*

*(1) If $(G, \alpha)$ has exactly one distinct edge label, then its degree sequence is $(1, 0, n-1)$.*
*(2) If $(G, \alpha)$ has exactly two distinct edge labels, then its degree sequence is $(1, 0, n-2, 0, 1)$.*
*(3) If $(G, \alpha)$ has three or more distinct edge labels, then its degree sequence is $(1, 0, n-3, 2)$.*

*Proof.* We prove each of (1)–(3) separately.

*Proof of* (1). This is an immediate consequence of Theorem 3.1: in the MGS $\{\mathbf{1}, \mathbf{I}^{v_2}, \ldots, \mathbf{I}^{v_{|V|}}\}$, the trivial spline $\mathbf{1}$ is a degree-zero spline and each of the $(|V| - 1)$-many $\mathbf{I}^{v_i}$ is a degree-two spline.

*Proof of* (2). The proof is essentially an analysis of the MGS $\mathcal{B}$ produced by Theorem 3.2 for a certain nice vertex ordering. Since $(G, \alpha)$ has exactly two distinct edge-labels, we choose an ordering of the vertices satisfying Proposition 2.1 by choosing the last vertex $v_n \in V$ to be any vertex incident to two edges with different labels; the vertex $v_1$ is chosen as the next vertex clockwise from $v_n$, and we continue choosing vertices $v_2, \ldots, v_{n-1}$ clockwise until all vertices have been ordered. Without loss of generality, suppose that $\alpha(v_{n-1}v_n) = \langle \mathsf{b} \rangle$ and $\alpha(v_nv_1) = \langle \mathsf{a} \rangle$.

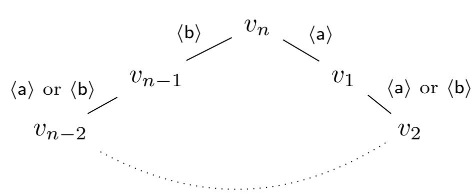



FIGURE 2. The idea of the proof of (2).

We claim that

- $\mathbf{1}$ is a degree-zero spline,
- $\mathbf{b}^i$ is a degree-two spline for all $2 \le i < n$, and
- $\mathbf{b}^n$ is a degree-four spline.

The first assertion is clear. For the second assertion, let us assume that while producing $\mathbf{b}^i$ using Theorem 3.2, we chose $v_j = v_{i-1}$.

Case 1: The edge-label $\alpha(v_iv_{i-1}) = \langle \mathsf{a} \rangle$.
Case 2: The edge-label $\alpha(v_iv_{i-1}) = \langle \mathsf{b} \rangle$.

The graph $C = (V', E')$ has vertex set $V'$ a subset of the set $\{v_i, v_{i+1}, \ldots, v_n\}$ in Case 1 (resp. $\{v_i, v_{i+1}, \ldots, v_{n-1}\}$ in Case 2). In both cases, Theorem 3.2 (a) must have been applied in the production of $\mathbf{b}^i$, so the spline $\mathbf{b}^i$ is a degree-two spline, and we have verified the second assertion.

For the third assertion, we again assume that while producing $\mathbf{b}^n$ using Theorem 3.2, we chose $v_j = v_{i-1} = v_{n-1}$. Now the graph $C$ contains the edge $v_nv_1$, so Theorem 3.2 (b) must have been applied in the production of $\mathbf{b}^n$. Hence $\mathbf{b}^n$ is a degree-four spline, and we have verified the third and final assertion.

*Proof of* (3). This is a consequence of Theorem 4.15. By Corollary 4.12, the homogeneous MGS $\mathcal{B}_{n-k}$ for $R_{C_{n-k},\alpha_{n-k}}$ has degree sequence $(1,0,n-k-3,2)$. For every $0 \leq j \leq k-1$, the homogeneous MGS $\mathcal{B}_{n-k+(j+1)}$ for $R_{C_{n-k+(j+1)},\alpha_{n-k+(j+1)}}$ has degree sequence $(1,0,n-k+(j+1)-3,2)$. After all iterations (when $j=k-1$), we obtain the homogeneous MGS $\mathcal{B}_n$ with degree sequence $(1,0,n-3,2)$ as desired. □

**Example 4.17.** We produce a homogeneous MGS $\mathcal{B}_6$ for the following edge-labeled six-cycle by using the algorithm in Theorem 4.15.

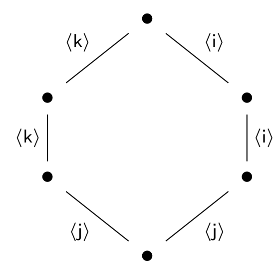



FIGURE 3. An edge-labeled six-cycle.

In Figure 4 below, we show one picture for each iteration through Steps (a)–(c) in Theorem 4.15 along with the associated homogeneous MGS obtained during that iteration.

## 5. Applications

The purpose of this section is to connect the theoretical apparatus of the previous parts of the paper to classical questions in the study of analytic splines. We show that our hypothesis on the homogeneity of polynomial splines in Section 4 is a very common one. In fact, it is satisfied by all graphs arising from GKM constructions of equivariant cohomology (see Proposition 5.2) and by most classical (analytic) applications of splines (see Remark 5.10). We then use our results to recover well-known results that classify splines on "pinwheel" triangulations in the plane.

5.1. **Splines and GKM theory.** One important application of the class of splines to which the results of this paper apply is the equivariant cohomology rings that arise from so-called *GKM theory*. In the rest of this section, we describe these rings.

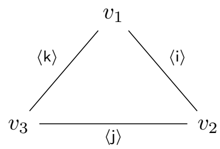
GKM theory describes an algebraic-combinatorial construction of the torus-equivariant cohomology of certain algebraic varieties. GKM theory applies to any complex projective algebraic variety $X$ that admits the action of a complex torus $T \cong (\mathbb{C}^*)^n$ with the following conditions (sometimes called the *GKM conditions*):

- There are finitely many $T$-fixed points in $X$.
- There are finitely many one-dimensional $T$-orbits in $X$.
- The variety $X$ is *equivariantly formal* with respect to the $T$-action.

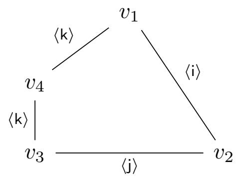
We omit a precise definition of *equivariantly formal*, which can be found elsewhere in the literature [GKM98, Tym05]. However, we note that the condition is satisfied by many varieties of interest, including varieties with no odd-dimensional ordinary cohomology. Moreover, equivariant formality implies that the equivariant cohomology $H_T^*(X)$ is a free $\mathbb{C}[t_1,\ldots,t_n]$-module. In fact, there is an isomorphism

$$H_T^*(X) \cong \mathbb{C}[t_1,\ldots,t_n] \otimes H^*(X).$$

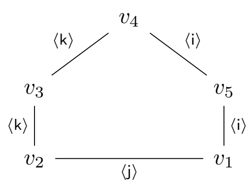
(The variables $t_1,\ldots,t_n$ denote the linearized coordinates of the torus, and are equivalent to the $x_1,\ldots,x_n$ in the polynomial splines of this paper. Informally, the weight of the torus action on a



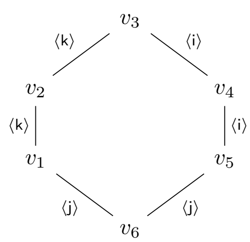

$$\mathcal{B}_3 = \left\{ \mathbf{1}, \begin{pmatrix} 0 \\ x\mathsf{i} \\ g_1\mathsf{k} \end{pmatrix}, \begin{pmatrix} 0 \\ y\mathsf{i} \\ g_2\mathsf{k} \end{pmatrix} \right\}$$




$$\mathcal{B}_4 = \left\{ \mathbf{1}, \begin{pmatrix} 0 \\ x\mathsf{i} \\ g_1\mathsf{k} \\ g_1\mathsf{k} \end{pmatrix}, \begin{pmatrix} 0 \\ y\mathsf{i} \\ g_2\mathsf{k} \\ g_2\mathsf{k} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathsf{k} \end{pmatrix} \right\}$$




$$\mathcal{B}_5 = \left\{ \mathbf{1}, \begin{pmatrix} x\mathsf{i} \\ g_1\mathsf{k} \\ g_1\mathsf{k} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} y\mathsf{i} \\ g_2\mathsf{k} \\ g_2\mathsf{k} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \mathsf{k} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mathsf{i} \end{pmatrix} \right\}$$




$$\mathcal{B}_6 = \left\{ \mathbf{1}, \begin{pmatrix} g_1\mathsf{k} \\ g_1\mathsf{k} \\ 0 \\ 0 \\ x\mathsf{i} \\ x\mathsf{i} \end{pmatrix}, \begin{pmatrix} g_2\mathsf{k} \\ g_2\mathsf{k} \\ 0 \\ 0 \\ y\mathsf{i} \\ y\mathsf{i} \end{pmatrix}, \begin{pmatrix} 0 \\ \mathsf{k} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathsf{i} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mathsf{j} \end{pmatrix} \right\}$$

FIGURE 4. An illustration of the algorithm in the proof of Theorem 4.15.

one-dimensional orbit is the "direction" of the torus flow in that orbit, and is given by a line in the $t_i$.)

GKM theory was developed without the language of splines, so we rephrase the key result with our terminology.

**Theorem 5.1** (Goresky–Kottwitz–MacPherson [GKM98]). *Suppose $T$ is a complex algebraic torus that acts on the complex projective algebraic variety $X$ satisfying the GKM conditions. Let $G_X$ be the graph with vertex set equal to the set of $T$-fixed points and containing an edge $uv$ exactly when $u, v$ are the two $T$-fixed points in the closure of a one-dimensional $T$-orbit on $X$. Furthermore, let $\alpha_X$ be the edge-labeling with $\alpha_X(uv)$ equal to the principal ideal generated by the weight at $u$ of the torus action on the one-dimensional orbit associated to $uv$. Then we have the following isomorphism, both as rings and as $\mathbb{C}[t_1, \ldots, t_n]$-modules:*

$$H_T^*(X) \cong R_{G_X, \alpha_X}.$$

The graph $G_X$ is sometimes called the *moment graph* of $X$ because it is the 1-skeleton of the moment polytope of $X$ in the case when $X$ is a symplectic manifold with a Hamiltonian $T$-action.

In fact, the torus-weights for varieties $X$ satisfying our conditions will all be homogeneous linear forms in the $t_i$. In particular, all edge-labels are principal ideals generated by homogeneous polynomials. We state this fact precisely for future reference.

**Proposition 5.2.** *All edge-labeled graphs $(G_X, \alpha_X)$ that arise from GKM constructions have principal ideals generated by homogeneous polynomials of degree one.*

The next section describes certain conditions under which restricting the degrees of splines over polynomial rings is equivalent to considering the same splines over quotient rings. A different quotient construction arises in GKM theory when constructing ordinary cohomology from equivariant cohomology, to which the next section does not apply.

5.2. **Splines with quotient rings as coefficients.** We now describe how the quotient map $R \to R/I$ on the coefficient ring affects splines, especially when $R$ is a polynomial ring, when the edge-labels are principal ideals generated by homogeneous elements, and when the ideal $I$ is generated by all monomials of a fixed degree. The key point is that in this case, the quotient is equivalent to restricting degree on the original collection of splines, in a way we make precise. Splines of degree at most $d$ will arise in the next section as the vector space $S^r_d(\Delta)$ of classical splines. The work in this section endows $S^r_d(\Delta)$ with part of the product structure of the ring of splines.

The first proposition describes the general relationship between splines over a ring $R$ and splines over a quotient $R/I$ and then specifies to the case when $R$ is the ring of polynomials and $I$ is the ideal generated by all monomials of a specific degree. The underlying strategy is similar to that of Bowden and the third author [BT15, Theorem 3.7].

**Proposition 5.3.** *Let $R$ be a ring with ideal $I$. Suppose that $(G, \alpha)$ is an edge-labeled graph over $R$, and let $(G, \overline{\alpha})$ be the edge-labeled graph over the ring $R/I$ obtained by composing the edge-labeling $\alpha$ with the usual quotient map $R \to R/I$. Define a map*

$$\pi\colon R_{G,\alpha} \to R_{G,\overline{\alpha}}$$

*by letting the image $\pi(p)$ be the spline $\overline{p}\colon V \to R/I_{d+1}$ that composes $p\colon V \to R$ with the quotient map $R \to R/I_{d+1}$. Then $\pi$ sends the ring of splines $R_{G,\alpha}$ over coefficient ring $R$ to the splines $R_{G,\overline{\alpha}}$ over the ring $R/I$.*

*Suppose $R = \mathbb{C}[x_1, \ldots, x_n]$ and $I = I_{d+1}$ is the ideal generated by monomials of degree $d+1$. Let $k$ be a nonnegative integer so that for every edge $uv \in E$ the edge-label $\alpha(uv)$ is a principal ideal generated by a homogeneous polynomial of degree $k$. Let $\mathcal{S}_d \subseteq R_{G,\alpha}$ be the subset of splines of degree at most $d$. Then $\pi\colon \mathcal{S}_d \to R_{G,\overline{\alpha}}$ is an isomorphism of complex vector spaces. Furthermore, if $p_1, p_2 \in \mathcal{S}_d$ satisfy $p_1p_2 \in \mathcal{S}_d$ then $\pi(p_1p_2) = \pi(p_1)\pi(p_2)$.*

*Proof.* The map $\pi$ is well-defined because it is a composition of well-defined maps.

We choose a unique polynomial lift for each coset in $\mathbb{C}[x_1, \ldots, x_n]/I_{d+1}$ so that if two cosets satisfy the spline condition over $\mathbb{C}[x_1, \ldots, x_n]/I_{d+1}$ then their polynomial lifts satisfy the spline condition as polynomials. Given a coset $\overline{p} \in \mathbb{C}[x_1, \ldots, x_n]/I_{d+1}$, define the polynomial $p$ as the sum of all terms of degree at most $d$ in any polynomial representative of $\overline{p}$. Note that $p$ is itself the polynomial representative of $\overline{p}$ with fewest terms.

Two polynomials $\overline{p}_u$ and $\overline{p}_v$ satisfy the spline condition on the edge $uv$ over the coefficient ring $\mathbb{C}[x_1, \ldots, x_n]/I_{d+1}$ exactly if

$$p_u - p_v = q(f + g_1) + g_2$$

for some $q \in \mathbb{C}[x_1, \ldots, x_n]$, homogeneous degree-$k$ generator $f \in \alpha(uv)$, and $g_1, g_2 \in I_{d+1}$. Decompose $q$ into its homogeneous parts as $q = q_0 + q_1 + \cdots + q_l$. Then $q(f + g_1) + g_2$ decomposes into homogeneous parts as

$$(q_0 + \cdots + q_{d-k})\, f + (q_0 + \cdots + q_{d-k})g_1 + (q_{d-k+1} + \cdots + q_l)(f + g_1) + g_2$$

Since the left summand is precisely the part of degree at most $d$, we conclude that

$$p_u - p_v = (q_0 + \cdots + q_{d-k})\, f$$

is in $\alpha(uv)$ as desired.

By construction of the lifts, the image $\pi(\mathcal{S}_d)$ is all of $R_{G,\overline{\alpha}}$. Furthermore $\pi$ is injective when restricted to $\mathcal{S}_d$ because the quotient map $\mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]/I_{d+1}$ is injective when restricted to polynomials of degree at most $d$. Finally the map $\pi$ is multiplicative when defined, again because the general quotient of rings $R \to R/I$ is a ring homomorphism, as is the product of quotients $R^{|V|} \to (R/I)^{|V|}$.

We have thus proven that $\pi \colon \mathcal{S}_d \to R_{G,\overline{\alpha}}$ is an isomorphism of complex vector spaces that preserves multiplication when the product is defined, as desired. □

**Remark 5.4.** We need the hypothesis of homogeneous generators for a $\mathbb{C}$-vector space isomorphism between splines of degree at most $d$ and the generalized splines in the quotient by $I_{d+1}$. Indeed, consider the graph in Figure 5. Consider the spline condition over the edge labeled $x^2 - 1$. When we

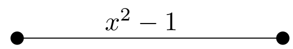



FIGURE 5. Example of an edge-labeled graph for which splines of bounded degree are not isomorphic to splines over coefficients in the quotient ring.

take coefficients in $\mathbb{C}[x]/I_2$, the edge-label becomes $-1$ and so all vertex-labelings satisfy the spline condition. However, zero is the only polynomial with degree at most one that is divisible by $x^2 - 1$. This means that only constant splines are in the image of the map from $\mathcal{S}_1$ to $R_{G,\overline{\alpha}}$.

The next result follows directly from Proposition 5.3, since generators project to generators under the quotient map.

**Corollary 5.5.** *Fix a finite integer $d \geq 0$, and let $I_{d+1} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be the ideal generated by all monomials of degree $d+1$. Let $(G, \alpha)$ be an edge-labeled graph satisfying the hypotheses of Proposition 5.3, and let $\mathcal{B}$ be a homogeneous MGS for $R_{G,\alpha}$. Then the degree sequence of $\overline{\mathcal{B}}$ for the spline module $R_{G,\overline{\alpha}}$ is the restriction to the first $d+1$ terms of the degree sequence of $\mathcal{B}$ for the spline module $R_{G,\alpha}$.*

In practice, we use the previous corollary to relate the degree sequence of $R_{G_\Delta, \alpha_\Delta^{r+1}}$ to the dimension of the classical space of splines, which we describe next.

5.3. **Classical results on splines.** Traditional splines are defined as piecewise polynomials on a particular form of geometric decomposition of a space (triangulation, polyhedral, etc.), usually restricted to degree at most $d$ and differentiability at least $r$. For our purposes, it is sufficient to take $\Delta$ to be a finite $n$-dimensional simplicial complex embedded in $\mathbb{R}^n$ with set of $n$-dimensional simplices $\{\sigma_v\}$. We view $\Delta$ as both a set of simplices and as a subset of $\mathbb{R}^n$, depending on context.

**Definition 5.6.** Let $r$ and $d$ be nonnegative integers. The *space of splines* $S_d^r(\Delta)$ is the $\mathbb{R}$-vector space defined by the property that $F \in S_d^r(\Delta)$ if and only if $F \colon \Delta \to \mathbb{R}$ is a function that

- has degree at most $d$ in the sense that each restriction $F|_{\sigma_v}$ is a polynomial of degree at most $d$, and
- is continuously differentiable of order $r$ as a function defined on a subspace of $\mathbb{R}^n$, namely $F$ is in $\mathcal{C}^r$.

The splines we consider elsewhere in this paper are a dualization of the classical formulation in Definition 5.6, as follows.

**Definition 5.7.** Suppose $\Delta$ is an $n$-dimensional simplicial complex with $n$-simplices $\{\sigma_v\}$.

- The *dual graph* $G_\Delta$ is the graph whose vertex set $V_\Delta$ is indexed by the collection of $n$-dimensional simplices $\sigma_v \in \Delta$ and whose edge set $E_\Delta$ contains the edge $uv$ whenever the corresponding $n$-simplices intersect in an $(n-1)$-dimensional simplex $\sigma_u \cap \sigma_v$.
- The *dual edge-labeling* $\alpha_\Delta$ is the edge-labeling in which $uv$ is labeled by the principal ideal $\alpha_\Delta(uv)$ generated by any nonzero *affine* linear form that vanishes on $\sigma_u \cap \sigma_v$.
- The *dual map* is a map from splines in $S_d^r(\Delta)$ to functions on the vertex set $V_\Delta$ of $G_\Delta$. If $F \in S_d^r(\Delta)$, then the dual map sends $F$ to the function $F^* \colon V_\Delta \to \mathbb{C}[x_1, \ldots, x_n]$ defined by $F^*(v) = F|_{\sigma_v}$ for all vertices $v$.

Note that $G_\Delta$ is the classical graph dual to $\Delta$ and that the dual edge-labeling $\alpha_\Delta$ is the function that assigns to each edge $uv$ in $G_\Delta$ the ideal generated by the equation of the line at the intersection of the triangles corresponding to $u$ and $v$.

Billera proved that the dual map is actually an isomorphism of vector spaces between classical splines and splines as defined in Definition 2.2. In other words, the splines used in this paper are a kind of dualization of classical splines. We describe Billera's result in Proposition 5.8 below. We do not give a formal definition of *simplex*, *strongly connected*, or *link* and instead refer the interested reader to either [Bil88] or any introductory text on polytopes (see, for example, [Zie95]). Recall also that for principal ideals, a power of an ideal is equal to the ideal generated by that power of the generator.

**Proposition 5.8** (Billera [Bil88, Theorem 2.4])**.** *Suppose $\Delta$ is a strongly-connected $n$-dimensional simplicial complex so that the link of each simplex in $\Delta$ is also strongly connected.*

*Define the $(r+1)^{th}$-power of $\alpha_\Delta$ to be the edge-labeling $\alpha_\Delta^{r+1}$ that associates to each edge $uv$ the ideal $(\alpha_\Delta(uv))^{r+1}$. Consider the module of generalized splines $R_{G_\Delta,\alpha_\Delta^{r+1}}$ with coefficients in the polynomial ring $R=\mathbb{C}[x_1,\ldots,x_n]$.*

*Then $F \in S_d^r(\Delta)$ if and only if $F^* \in R_{G_\Delta,\alpha_\Delta^{r+1}}$ is a spline whose localizations $F^*(v)$ have degree at most $d$ for all $v \in V_\Delta$. As vector spaces, this dual map is an isomorphism.*

The hypotheses in the first sentence of Proposition 5.8 are satisfied by most decompositions that arise in applications. For instance, suppose the simplicial complex $\Delta$ is a triangulation of a region in the plane. In this case, that the link of a vertex is strongly connected means that there are no "pinch points" in the region; that is, the region's boundary is a disjoint union of subspaces homotopic to circles.

Generalized splines are strictly more general than classical splines in various ways: dual graphs to triangulations must have trivalent interior vertices (or other regularity conditions on most vertices, in the case of more general simplices); indeed, dual graphs to planar graphs are planar. On the algebraic side, the ideals that arise as the image of $\alpha_\Delta$ must be principal (unlike most ideals), and the underlying ring is a polynomial ring or quotient thereof (unlike most rings).

We now specialize Proposition 5.3 to splines on dual graphs, using Billera's result from Proposition 5.8 to show the (classical) space of splines $S_d^r(\Delta)$ is the space of generalized splines $R_{G_\Delta,(\overline{\alpha_\Delta})^{r+1}}$ over the quotient ring. Note that this endows $S_d^r(\Delta)$ with a product structure.

**Corollary 5.9.** *Assume $\Delta$ satisfies the hypotheses of Proposition 5.8. Suppose that every ideal $\alpha(uv)$ is a principal ideal generated by a homogeneous element in $\mathbb{C}[x_1,\ldots,x_n]$, and let $I_{d+1}$ be the ideal in $\mathbb{C}[x_1,\ldots,x_n]$ generated by all homogeneous elements of degree $d+1$. Let $R_{G_\Delta,(\overline{\alpha_\Delta})^{r+1}}$ denote the dual ring of splines over the quotient ring.*

*Then $F \in S_d^r(\Delta)$ if and only if $\overline{F^*} \in R_{G_\Delta,(\overline{\alpha_\Delta})^{r+1}}$, where $\overline{F^*}$ is $F^*$ composed with the quotient map $\mathbb{C}[x_1,\ldots,x_n] \to \mathbb{C}[x_1,\ldots,x_n]/I_{d+1}$. The map $F \to \overline{F^*}$ is a $\mathbb{C}$-vector space isomorphism and respects multiplication in the following senses:*

- *If $f \in \mathbb{C}[x_1,\ldots,x_n]$ and $p \in S_d^r(\Delta)$ satisfy $fp \in S_d^r(\Delta)$, then*
$$\overline{fp} = \overline{f}\,\overline{p} \in R_{G,(\overline{\alpha_\Delta})^{r+1}}.$$
- *If $p_1, p_2 \in S_d^r(\Delta)$ satisfy $p_1p_2 \in S_d^r(\Delta)$, then*
$$\overline{p_1p_2} = \overline{p_1}\,\overline{p_2} \in R_{G,(\overline{\alpha_\Delta})^{r+1}}.$$

*Proof.* Billera's original result proved that the dual map $F \to F^*$ is a well-defined bijection of vector spaces when $R_{G,\alpha_\Delta^{r+1}}$ is considered with underlying ring $\mathbb{C}[x_1,\ldots,x_n]$. Proposition 5.3 composes with the quotient map and completes the proof. □

**Remark 5.10.** Note that the edge-labeling of Figure 5 in Remark 5.4 cannot occur in a graph $G_\Delta$ dual to a standard triangulation $\Delta$ because all edge-labels $\alpha_\Delta(uv)$ have the form $(ax+by+c)^{r+1}$ for some $a,b,c \in \mathbb{C}$. Moreover, we can always homogenize an edge-labeling over a polynomial ring

$\mathbb{C}[x_1, \ldots, x_n]$, generally via an additional variable. In other words, Corollary 5.9 applies to all splines arising in classical (analytic) contexts and in applied mathematics.

The main tool in the final section is the following lemma, which establishes that generators for finite edge-labeled graphs $(G, \alpha)$ can be assumed to be in a certain kind of general position. More importantly, the lemma constrains the edge-labeled graphs $(G, \alpha)$ that can arise as the duals to a simplicial complex. In essence, it says that we may change coordinates to assume that any particular interior vertex in a simplicial complex is the origin of the plane, and then reinterprets that for the edge-labeling of the dual graph.

This lemma reinforces the point that generalized splines are more general than splines dual to simplicial complexes. It is not generally true that every edge-labeling can be modified by an affine linear operator so that any individual vertex is incident only to ideals generated by polynomials with nonconstant terms! This is a constraint on the graph inherited from the geometry when its dual space is embedded in Euclidean space.

**Lemma 5.11.** *Consider the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. Suppose that $(G, \alpha)$ is a finite graph for which each edge-label $\alpha(uv)$ is the principal ideal generated by*

$$(a_{1,uv}x_1 + a_{2,uv}x_2 + \cdots + a_{n,uv}x_n + c_{uv})^{r_{uv}}$$

*for some positive integer $r_{uv}$ and constants $a_{1,uv}, \ldots, a_{n,uv}, c_{uv} \in \mathbb{C}$.*

*For each vertex $v_0$, there is an edge-labeling $\alpha'$ so that*

*(1) all edge-labels $\alpha'(uv)$ are generated by polynomials with every coefficient $a'_{i,uv}$ nonzero,*
*(2) $\alpha'$ is constructed by composing $\alpha$ with a linear operator that acts as a rotation, and*
*(3)*
$$R_{G,\alpha} \cong R_{G,\alpha'}.$$

*Furthermore, suppose $\Delta$ is any simplicial complex that satisfies the hypotheses of Proposition 5.8, with dual graph $G_\Delta$ and dual edge-labeling $\alpha_\Delta^{r+1}$. Then $\alpha'$ can also be chosen to satisfy*

*(4) all edge-labels $\alpha'(uv_0)$ incident to $v_0$ are generated by homogeneous polynomials, and*
*(5) $\alpha'$ can be constructed by composing $\alpha_\Delta$ with the linear operator that translates the vertex $v_0$ to the origin and then performs a rotation of Euclidean space around the origin.*

*Proof.* Let $g$ be an invertible affine linear operator on the $\mathbb{C}$-vector space spanned by $x_1, x_2, \ldots, x_n$, and let $\alpha' = \alpha \circ g$. The map induced by $g$ on $\mathbb{C}[x_1, \ldots, x_n]$ is invertible and thus induces an algebra isomorphism

$$R_{G,\alpha} \cong R_{G,\alpha'}.$$

We now choose a particular rotation operator $g$ that satisfies Condition (1) from the claim, namely that each coefficient $a'_{i,uv}$ in the composition $\alpha \circ g$ is nonzero. The composition $\alpha \circ g$ sends the coefficient $a_{i,uv}$ to the expression

$$g_{i1}a_{1,uv} + g_{i2}a_{2,uv} + \cdots + g_{in}a_{n,uv}. \tag{9}$$

This expression is zero if and only if the $g_{ij}$ satisfy a particular linear equation, equivalently lie on a particular hyperplane. The condition that *none* of the expressions in Equation (9) be zero is equivalent to asking that the $g_{ij}$ avoid a finite set of hyperplanes. In particular, consider the matrices $g$ representing rotations around the origin. This is a unitary group and so intersects each hyperplane of Equation (9) in a subspace of codimension one. No finite union of these codimension-one subspaces can cover the entire space of possible $g$. This shows we may choose a rotation $g$ so that no coefficient $a'_{i,uv}$ is zero, which proves the first claim.

Finally, suppose $(G_\Delta, \alpha_\Delta)$ arises as the dual of a simplicial complex $\Delta$ satisfying the hypotheses of Proposition 5.8. The vertex $v_0$ is a point in $\mathbb{C}^n$. Let $g'$ be the translation of $\mathbb{C}^n$ that moves $v_0$ to the origin, namely that sends the vector $\mathbf{b}x$ to $\mathbf{b}x - v_0$. Thus $g'$ induces a change of coordinates on $\mathbb{C}[x_1, \ldots, x_n]$ with respect to which $v_0$ becomes the zero vector, so equations satisfied by $v_0$ must now be satisfied when all $x_i = 0$. In particular if $\alpha$ labels the edge $uv_0$ with the principal

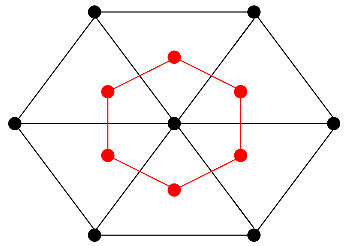


FIGURE 6. A pinwheel with its dual graph shown in red.

ideal generated by $(a_{1,uv_0}x_1 + a_{2,uv_0}x_2 + \cdots + a_{n,uv_0}x_n + c_{uv_0})^{r_{uv_0}}$, then $\alpha \circ g'$ labels $uv_0$ with the ideal generated by $(b_{1,uv_0}x_1 + b_{2,uv_0}x_2 + \cdots + b_{n,uv_0}x_n)^{r_{uv_0}}$ for some constants $b_{i,uv_0} \in \mathbb{C}$. In other words, the edge-labeling $\alpha \circ g'$ assigns homogeneous degree-one generators to all edges incident to $v_0$. Composing with a rotation operator as described previously completes the proof. □

5.4. **Applications to splines on planar triangulations.** In this final section, we apply earlier work in this paper to the lower bound formula described in the introduction. The main open questions about that formula address splines $S^1_d(\Delta)$ on a planar triangulation $\Delta$ for low degrees $d$. This is one reason we focused on quadratic edge-labels and low-degree splines in parts of this paper. Our main application characterizes splines on "pinwheels," recovering a result of Lai–Schumaker, and uses that to provide support for the lower bound formula.

Much of this section applies to the special case of an interior cell (or *pinwheel* triangulation), which is a triangulation that has a unique interior vertex around which triangles radiate like the spokes of a wheel. This is shown in Figure 6 together with its dual graph, which is simply a cycle.

**Theorem 5.12.** *Let $\Delta$ be the triangulation of an interior cell, namely a pinwheel with $n$ triangles, and let $I_{d+1} \subseteq \mathbb{C}[x, y]$ be the ideal generated by all monomials of degree $d+1$. Then the map $F \to F^*$ is an isomorphism of complex vector spaces between the splines $S^r_d(\Delta)$ and the generalized splines $R_{C_n,\overline{\alpha}}$, where $C_n$ is an $n$-cycle and $\alpha$ is an edge-labeling so that every ideal $\alpha(uv)$ is principal and generated by $(x + a_{uv}y)^{r+1}$ for a nonzero $a_{uv} \in \mathbb{C}^*$. Moreover, for each $a_{uv}$ there is at most one other edge $u'v'$ with $a_{u'v'} = a_{uv}$, and that edge cannot immediately follow or precede $uv$.*

*Proof.* Proposition 5.8 applies to the pinwheel triangulation with $n$ triangles, so $S^r_d(\Delta)$ is isomorphic to the generalized splines on the graph dual to $\Delta$ with edge-labeling given by the $(r+1)^{\text{th}}$ power of the equations of the lines through the central vertex in $\Delta$. The graph dual to a pinwheel triangulation with $n$ triangles is a cycle on $n$ vertices. By Lemma 5.11, we may identify the central vertex of the triangulation with the origin and assume each edge $uv$ is labeled by $(x + a_{uv}y)^{r+1}$ for nonzero coefficients $a_{uv}$. Finally, at most two rays through a given point lie on the same line, so no more than two of the edge-labels can coincide; if two successive rays going clockwise around the central vertex are the same, then the triangle they describe has more than 180° as its interior angle-sum, which is impossible. This proves the claim. □

We will use the previous theorem to reinterpret the main results of earlier sections. The key observation is the following, which characterizes cycles that can be realized as the dual of a triangulation.

**Lemma 5.13.** *All edge-labeled cycles $(C, \alpha)$ that are geometrically realizable as the dual of a triangulation must have at least three edge-labels unless the cycle is a four-cycle with two distinct edge-labels that alternate around the cycle.*

*Proof.* A cycle is dual to a triangulation only if that triangulation is an interior cell (namely pinwheel triangulation). Suppose $(C, \alpha)$ is dual to a triangulation. Theorem 5.12 implies that if $C$ has three edges, then they are all labeled distinctly; if $C$ has at least five edges, then at least three successive edges must be labeled distinctly. The only four-cycles with fewer than three distinct edge-labels are precisely those dual to pinwheel triangulations formed by the intersection of two lines. This gives an edge-labeled four-cycle that alternates between two distinct edge-labels as one moves around the cycle. □

Classically, interior vertices formed by the intersection of two lines play a special role in the theory of splines on triangulations. We give this terminology in the context of generalized splines on the dual graph.

**Definition 5.14.** The interior vertices of the triangulations corresponding to 2-label 4-cycles are called *singular vertices*.

Lemma 5.13 thus shows that singular vertices are special insofar as they correspond to the *only geometrically realizable* edge-labeled cycles with at most two distinct edge labels.

Combining these results with those from earlier sections gives an explicit algorithm for constructing a minimal generating set for splines on interior cells. The first consequence is a classical result for general $r$ and $d$ [LS07, Theorems 9.3 and 9.12].

**Corollary 5.15.** *Denote the number of monomials of degree at most $d$ by $m_d$, namely*

$$m_d = 1 + \cdots + (d+1) = \frac{(d+1)(d+2)}{2}.$$

*The dimension of the classical spline space $S^1_d(\Delta)$ of splines on a triangulation $\Delta$ corresponding to a pinwheel triangulation (or interior cell) with $n$ triangles has two formulas.*

*If the pinwheel has four triangles and a singular vertex, then the dimension of $S^1_d(\Delta)$ is*

$$\begin{cases} m_d & \text{if } d \leq 1, \\ m_d + 2m_{d-2} & \text{if } 2 \leq d \leq 3, \\ m_d + 2m_{d-2} + m_{d-4} & \text{if } d \geq 4. \end{cases}$$

*If the pinwheel has $n \geq 3$ triangles and no singular vertex, then the dimension of $S^1_d(\Delta)$ is*

$$\begin{cases} m_d & \text{if } d \leq 1, \\ m_d + (n-3)m_{d-2} & \text{if } d = 2, \\ m_d + (n-3)m_{d-2} + 2m_{d-3} & \text{if } d \geq 3. \end{cases}$$

*Proof.* Lemma 5.13 asserts that the pinwheel with four triangles and a singular vertex is the only geometrically-realizable cycle with just two distinct labels. Theorem 3.2 constructed an upper-triangular basis for the module of splines over the polynomial ring in the two-label case. Each module generator of degree $j$ contributes $m_{d-j}$ elements to the vector space basis in degree at most $d$, since each module generator can be multiplied by each of the $m_{d-j}$ monomials of degree at most $d-j$.

If $\Delta$ is not the pinwheel with a singular vertex, Theorem 5.12 showed that $\Delta$ must have at least three successive distinct labels. Lemma 4.14 gave a homogeneous basis for the spline space as a module over the polynomial ring in this case. Thus each module generator of degree $j$ contributes $m_{d-j}$ elements to the vector space basis in degree at most $d$. This proves the claim. □

These results also allow us to contextualize the lower bound conjecture described in the introduction. In particular, we can bound the dimension of $S^1_d(\Delta)$ by building the triangulation $\Delta$ one interior vertex at a time, and by using Corollary 5.15 to bound the contribution of each interior vertex.

**Corollary 5.16.** *Suppose $\Delta$ and $\Delta'$ are triangulations of a region in the plane satisfying the hypotheses of Proposition 5.8 and that $\Delta'$ is obtained by adding a new interior cell to $\Delta$ with $k$ triangles radiating around the new interior vertex.*

*Then the complex vector space $S^1_d(\Delta')$ may have more basis elements than $S^1_d(\Delta)$. The number of additional (vector space) basis elements is at most*

$$\dim\left(S^1_d(\Delta_0)\right) - m_d$$

*where $\Delta_0$ is the pinwheel triangulation with $k$ triangles and $m_d$ is the number of monomials of degree at most $d$.*

*Proof.* The preimage of the restriction map $R_{G_{\Delta'},\alpha_{\Delta'}} \to R_{G_\Delta,\alpha_\Delta}$ consists of the *nonconstant* splines in $R_{G_{\Delta_0},\alpha}$. The dimension of nonconstant splines is an upper bound on the total dimension of $R_{G_{\Delta'},\alpha_{\Delta'}}$ since the restriction might not be surjective. This dimension was given in Corollary 5.15, proving the claim. □

The condition that the link of a vertex is strongly connected in fact implies that any triangulation satisfying the constraints of Proposition 5.8 can be built one interior vertex at a time. We sketch the argument here. Since the link of each vertex is strongly connected, the link of each interior vertex is a cycle. If $\Delta'$ has an interior vertex, then there is an interior vertex lying on a triangle with a boundary edge. Removing this vertex and the triangles on which it lies leaves a triangulation $\Delta$. If $\Delta$ is connected, then it still satisfies the conditions of Proposition 5.8. For some choice of vertex $\Delta$ is connected, because $\Delta'$ is strongly connected.

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