LOGARITHMIC CONCAVITY OF SCHUR AND RELATED POLYNOMIALS

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ABSTRACT. We show that normalized Schur polynomials are strongly log-concave. As a consequence, we obtain Okounkov's log-concavity conjecture for Littlewood–Richardson coefficients in the special case of Kostka numbers.

1. INTRODUCTION

Schur polynomials are the characters of finite-dimensional irreducible polynomial representations of the general linear group $\mathrm{GL}_m(\mathbb{C})$. Combinatorially, the *Schur polynomial* of a partition λ in m variables is the generating function

$$s_\lambda(x_1,\ldots,x_m)=\sum_{\mathrm{T}} x^{\mu(\mathrm{T})}, \quad x^{\mu(\mathrm{T})}=x_1^{\mu_1(\mathrm{T})}\cdots x_m^{\mu_m(\mathrm{T})},$$

where the sum is over all Young tableaux T of shape λ with entries from $[m]$, and

$$\mu_i(\mathrm{T}) = \text{the number of } i\text{'s among the entries of } \mathrm{T}, \ \text{ for } i=1,\ldots,m.$$

Collecting Young tableaux of the same weight together, we get

$$s_\lambda(x_1,\ldots,x_m)=\sum_\mu K_{\lambda\mu}x^\mu,$$

where $K_{\lambda\mu}$ is the *Kostka number* counting Young tableaux of given shape λ and weight μ [Kos82]. Correspondingly, the *Schur module* $\mathrm{V}(\lambda)$, an irreducible representation of the general linear group with highest weight λ, has the weight space decomposition

$$\mathrm{V}(\lambda)=\bigoplus_\mu \mathrm{V}(\lambda)_\mu \ \text{ with } \ \dim \mathrm{V}(\lambda)_\mu = K_{\lambda\mu}.$$

Schur polynomials were first studied by Cauchy [Cau15], who defined them as ratios of alternants. The connection to the representation theory of $\mathrm{GL}_m(\mathbb{C})$ was found by Schur [Sch01]. For a gentle introduction to these remarkable polynomials, and for all undefined terms, we refer to [Ful97].

June Huh received support from NSF Grant DMS-1638352 and the Ellentuck Fund. Jacob Matherne received support from NSF Grant DMS-1638352 and the Association of Members of the Institute for Advanced Study. Karola Mészáros received support from NSF Grant DMS-1501059, CAREER NSF Grant DMS-1847284 and a von Neumann Fellowship funded by the Friends of the Institute for Advanced Study.

We prove several log-concavity properties of Schur polynomials. An operator that turns generating functions into exponential generating functions will play an important role. This linear operator, denoted N, is defined by the condition

$$\mathrm{N}(x^\mu) = \frac{x^\mu}{\mu!} = \frac{x_1^{\mu_1}}{\mu_1!} \cdots \frac{x_m^{\mu_m}}{\mu_m!} \ \text{ for all } \mu \in \mathbb{N}^m.$$

Recall that a *partition* is a weakly decreasing sequence of nonnegative integers.

Theorem 1 (Continuous)**.** For any partition λ, the normalized Schur polynomial

$$\mathrm{N}(s_\lambda(x_1, \ldots, x_m)) = \sum_\mu K_{\lambda\mu} \frac{x^\mu}{\mu!}$$

is either identically zero or its logarithm is concave on the positive orthant $\mathbb{R}^m_{>0}$.

Let e_i be the i-th standard unit vector in $\mathbb{N}^m$. For $\mu \in \mathbb{Z}^m$ and distinct $i, j \in [m]$, we set

$$\mu(i,j) = \mu + e_i - e_j.$$

We show that the sequence of weight multiplicities of $\mathrm{V}(\lambda)$ we encounter is always log-concave if we walk in the weight diagram along any root direction $e_i - e_j$.

Theorem 2 (Discrete)**.** For any partition λ and any $\mu \in \mathbb{N}^m$, we have

$$K_{\lambda\mu}^2 \geqslant K_{\lambda\mu(i,j)} K_{\lambda\mu(j,i)} \ \text{ for any } i, j \in [m].$$

For partitions ν, κ, λ, the *Littlewood–Richardson coefficient* $c^\nu_{\kappa\lambda}$ is given by the decomposition

$$\mathrm{V}(\kappa) \otimes \mathrm{V}(\lambda) \simeq \bigoplus_\nu \mathrm{V}(\nu)^{\oplus c^\nu_{\kappa\lambda}}.$$

When the skew shape ν/κ has at most one box in each column, $c^\nu_{\kappa\lambda}$ is the Kostka number $\mathrm{K}_{\lambda\mu}$, where $\mu = \nu - \kappa$.[1] Conversely, for any partition λ and any μ, we have

$$K_{\lambda\mu} = c^\nu_{\kappa\lambda},$$

where ν and κ are the partitions given by $\nu_i = \sum_{j=i}^n \mu_j$ and $\kappa_i = \sum_{j=i+1}^n \mu_j$. Thus Theorem 2 verifies a special case of Okounkov's conjecture that the discrete function

$$(\nu, \kappa, \lambda) \longmapsto \log c^\nu_{\kappa\lambda}$$

[1] The equality between the Littlewood–Richardson coefficient and the Kostka number follows from Pieri's formula

$$h_{\mu_1}(x_1, \ldots, x_m) \cdots h_{\mu_m}(x_1, \ldots, x_m) = \sum_\lambda K_{\lambda\mu} s_\lambda(x_1, \ldots, x_m),$$

where h_{μ_i} is the μ_i-th complete symmetric function [Ful97, Section 6.1]. When ν/κ has at most one box in each column, the left-hand side is the skew Schur function $s_{\nu/\kappa}$, given by the Littlewood–Richardson rule

$$s_{\nu/\kappa}(x_1, \ldots, x_m) = \sum_\lambda c^\nu_{\kappa\lambda} s_\lambda(x_1, \ldots, x_m).$$

is concave [Oko03, Conjecture 1].[2]

We point out that, for any fixed λ, the log-concavity of $K_{\lambda\mu}$ along *any* direction is known to hold *asymptotically*. By [Hec82], the *Duistermaat–Heckman measure* obtained from the orbit of λ under SU_m is a translate of the weak limit

$$\lim_{k\to\infty} \frac{\sum_\mu K_{k\lambda\mu}\delta_{\frac{1}{k}\mu}}{\sum_\mu K_{k\lambda\mu}},$$

where $\delta_{\frac{1}{k}\mu}$ is the point mass at $\frac{1}{k}\mu$. It follows from [Gra96] that, in this case, the density function of the Duistermaat–Heckman measure is log-concave.[3] We refer to [BGR04, Section 3] for an exposition.

In [BH19], the authors introduce Lorentzian polynomials as a generalization of volume polynomials in algebraic geometry and stable polynomials in optimization theory. See Section 2 for a brief introduction. We show that normalized Schur polynomials are Lorentzian in the sense of [BH19], and deduce Theorems 1 and 2 from the Lorentzian property.

Theorem 3. The normalized Schur polynomial $\mathrm{N}(s_\lambda(x_1,\dots,x_m))$ is Lorentzian for any λ.

Using general properties of Lorentzian polynomials [BH19, Section 6], Theorem 3 can be strengthened as follows.

Corollary 4. For any sequence of partitions $\lambda^1,\dots,\lambda^\ell$ and any positive integers $m_1,\dots,m_\ell$,

(1) the normalized product of Schur polynomials $\mathrm{N}(\prod_{k=1}^\ell s_{\lambda^k}(x_1,\dots,x_{m_k}))$ is Lorentzian, and

(2) the product of normalized Schur polynomials $\prod_{k=1}^\ell \mathrm{N}(s_{\lambda^k}(x_1,\dots,x_{m_k}))$ is Lorentzian.

We prove Theorem 3 in Section 2 in a more general context of Schubert polynomials, but the main idea is simple enough to be outlined here. The *volume polynomial* of an irreducible complex projective variety Y, with respect to a sequence of nef divisor classes[4] $\mathrm{H} = (\mathrm{H}_1,\dots,\mathrm{H}_m)$, is the homogeneous polynomial

$$\mathrm{vol}_{Y,\mathrm{H}}(x_1,\dots,x_m) = \frac{1}{\dim Y!}\int_Y (x_1\mathrm{H}_1+\cdots+x_m\mathrm{H}_m)^{\dim Y},$$

where the intersection product of Y is used to expand the integrand. Volume polynomials are prototypical examples of Lorentzian polynomials [BH19, Section 10]. To show that the normalized Schur polynomial of λ is a volume polynomial, we suppose that the partition λ has m parts,

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[2] The conjecture holds in the "classical limit" [Oko03, Section 3], but the general case is refuted in [CDW07]:

$$c^{(4^n,3^n,2^n,1^n)}_{(3^n,2^n,1^n)(2^n,1^n,1^n)} = \binom{n+2}{2} \text{ and } c^{(8^n,6^n,4^n,2^n)}_{(6^n,4^n,2^n)(4^n,2^n,2^n)} = \binom{n+5}{5} \text{ for all } n.$$

The same example shows that the log-concavity conjecture for *parabolic Kostka numbers* [Kir04, Conjecture 6.17] also fails.

[3] Let (M,ω) be a symplectic manifold of dimension $2n$ with an action of a torus T and a moment map $\mathrm{M}\to\mathfrak{t}^*$. The Duistermaat–Heckman measure is the push-forward of the Liouville measure $\int\omega^n$ via the moment map. In this generality, Karshon shows that the density function need not be log-concave [Kar96].

[4] A Cartier divisor on a complete variety Y is *nef* if it intersects every curve in Y nonnegatively. We refer to [Laz04] for a comprehensive introduction.

and choose a large integer ℓ to get a complementary pair of partitions

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \ \text{ and } \ \kappa = (\ell, \ell, \ldots, \ell) - (\lambda_m, \lambda_{m-1}, \ldots, \lambda_1).$$

The Schur polynomials of the partitions λ and κ are related by the identity[5]

$$s_\kappa(x_1, \ldots, x_m) = x_1^\ell \cdots x_m^\ell s_\lambda(x_1^{-1}, \ldots, x_m^{-1}).$$

Let X be the product of projective spaces $(\mathbb{P}^\ell)^m$, and let Y be a subvariety of X whose fundamental class satisfies

$$[Y] = s_\kappa(\mathrm{H}_1, \ldots, \mathrm{H}_m) \cap [X], \quad \mathrm{H}_i = c_1(\pi_i^* \mathcal{O}(1)),$$

where π_i is the i-th projection. The volume polynomial of Y with respect to H is

$$\begin{aligned}\mathrm{vol}_{Y,\mathrm{H}}(x_1, \ldots, x_m) &= \frac{1}{\dim Y!} \int_Y (x_1 \mathrm{H}_1 + \cdots + x_n \mathrm{H}_m)^{\dim Y} \\ &= \frac{1}{\dim Y!} \int_X s_\kappa(\mathrm{H}_1, \ldots, \mathrm{H}_m)(x_1 \mathrm{H}_1 + \cdots + x_m \mathrm{H}_m)^{\dim Y} = \mathrm{N}(s_\lambda(x_1, \ldots, x_m)).\end{aligned}$$

Such Y can be constructed from a sequence of generic global sections $\bigoplus_{i=1}^m \pi_i^* \mathcal{O}(1)$ as a degeneracy locus [Ful98, Example 14.3.2], completing the argument.

In Section 2, we introduce Lorentzian polynomials and prove the main results. In Section 3, we present evidence for the ubiquity of Lorentzian polynomials through a series of results and conjectures.

Acknowledgments. We are grateful to Dave Anderson, Alex Fink, Allen Knutson, Thomas Lam, Ricky Liu, Alex Postnikov, Pavlo Pylyavskyy, Vic Reiner, Mark Shimozono, and Alex Yong for fruitful discussions. We thank the Institute for Advanced Study for providing an excellent environment for our collaboration.

2. Normalized Schur polynomials are Lorentzian

A subset $\mathrm{J} \subseteq \mathbb{Z}^n$ is M-*convex*[6] if, for any index $i \in [n]$ and any $\alpha \in \mathrm{J}$ and $\beta \in \mathrm{J}$ whose i-th coordinates satisfy $\alpha_i > \beta_i$, there is an index $j \in [n]$ satisfying

$$\alpha_j < \beta_j \ \text{ and } \ \alpha - e_i + e_j \in \mathrm{J} \ \text{ and } \ \beta - e_j + e_i \in \mathrm{J}.$$

The notion of M-convexity forms the foundation of discrete convex analysis [Mur03]. The convex hull of an M-convex set is a *generalized permutohedron* in the sense of [Pos09], and conversely, the set of integral points in an integral generalized permutohedron is an M-convex set [Mur03, Theorem 1.9].

Lorentzian polynomials connect discrete convex analysis with many log-concavity phenomena in combinatorics. See [AOGV18, ALOGV18a, ALOGV18b, BES19, BH18, BH19, EH19] for

[5] The dual of the Schur module $\mathrm{V}(\lambda)$ has highest weight $(-\lambda_m, \ldots, -\lambda_1)$, see [FH91, Exercise 15.50].

[6] The letter M stands for *matroids*. When $\mathrm{J} \subseteq \mathbb{N}^n$ consists of zero-one vectors, the M-convexity of J is the symmetric basis exchange property of matroids [Whi86, Chapter 4].

recent applications. Here we briefly summarize the relevant results, and refer to [BH19] for details. We fix integers d and $e = d - 2$.

Definition 5. Let $h(x_1, \ldots, x_n)$ be a degree d homogeneous polynomial. We say that h is *strictly Lorentzian* if all the coefficients of h are positive and

$$\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_e}} h \text{ has the signature } (+, -, \ldots, -) \text{ for any } i_1, \ldots, i_e \in [n].$$

We say that h is *Lorentzian* if it satisfies any one of the following equivalent conditions.

(1) All the coefficients of h are nonnegative, the support of h is M-convex,[7] and

$$\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_e}} h \text{ has at most one positive eigenvalue for any } i_1, \ldots, i_e \in [n].$$

(2) All the coefficients of h are nonnegative and, for any $i_1, i_2, \ldots \in [n]$ and any positive k,

$$\text{the functions } h \text{ and } \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}} h \text{ are either identically zero or log-concave on } \mathbb{R}^n_{>0}.$$

(3) The polynomial h is a limit of strictly Lorentzian polynomials.

For example, a bivariate polynomial $\sum_{k=0}^{d} a_k x_1^k x_2^{d-k}$ with nonnegative coefficients is Lorentzian if and only if the sequence $a_0, \ldots, a_d$ has no internal zeros[8] and

$$\frac{a_k^2}{\binom{d}{k}^2} \geqslant \frac{a_{k-1}}{\binom{d}{k-1}} \frac{a_{k+1}}{\binom{d}{k+1}} \quad \text{for all } 0 < k < d.$$

Polynomials satisfying the second condition of Definition 5, introduced by Gurvits in [Gur09], are called *strongly log-concave*. See [BH19, Section 5] for a proof of the equivalence of the three conditions in Definition 5.

We write $\mathcal{S}_n$ for the group of permutations of $[n]$. The *Schubert polynomial* $\mathfrak{S}_w(x_1, \ldots, x_n)$ for $w \in \mathcal{S}_n$ can be defined recursively as follows.

(1) If $w = w_\circ$ is the longest permutation $n \; n-1 \; \cdots \; 2 \; 1$, then

$$\mathfrak{S}_w(x_1, \ldots, x_n) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1.$$

(2) If $w(i) > w(i+1)$ for some i and s_i is the adjacent transposition $(i \; i+1)$, then

$$\mathfrak{S}_{ws_i}(x_1, \ldots, x_n) = \partial_i \mathfrak{S}_w(x_1, \ldots, x_n).$$

The symbol ∂_i stands for the *i-th divided difference operator* defined by the formula

$$\partial_i \mathfrak{S}_w = \frac{\mathfrak{S}_w - s_i \mathfrak{S}_w}{x_i - x_{i+1}},$$

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[7]The *support* of a polynomial $h(x_1, \ldots, x_n)$ is the set of monomials appearing in h, viewed as a subset of $\mathbb{N}^n$.

[8]The sequence $a_0, \ldots, a_d$ has *no internal zeros* if $a_{k_1} a_{k_3} \neq 0 \Longrightarrow a_{k_2} \neq 0$ for all $0 \leqslant k_1 < k_2 < k_3 \leqslant d$.

where $s_i\mathfrak{S}_w$ is the polynomial obtained from $\mathfrak{S}_w$ by interchanging x_i and x_{i+1}. The divided difference operators satisfy the *braid relations*, and it follows that the Schubert polynomials are well-defined [MS05, Exercise 15.3]. For any $w \in \mathcal{S}_n$, we define

$$\mathfrak{S}_w^\vee = \mathrm{N}(x_1^{n-1}\cdots x_n^{n-1}\mathfrak{S}_w(x_1^{-1},\ldots,x_n^{-1})).$$

Theorem 6. The polynomial $\mathfrak{S}_w^\vee(x_1,\ldots,x_n)$ is Lorentzian for any $w \in \mathcal{S}_n$.

We conjecture that $\mathrm{N}(\mathfrak{S}_w(x_1,\ldots,x_n))$ is Lorentzian for any $w \in \mathcal{S}_n$, see Section 3.2.

Proof. Recall that the volume polynomial of a projective variety Y, with respect to a sequence of Cartier divisor classes $\mathrm{H} = (\mathrm{H}_1,\ldots,\mathrm{H}_n)$, is the homogeneous polynomial

$$\mathrm{vol}_{Y,\mathrm{H}}(x_1,\ldots,x_n) = \frac{1}{\dim Y!}\int_Y (x_1\mathrm{H}_1+\cdots+x_n\mathrm{H}_n)^{\dim Y}.$$

By [BH19, Theorem 10.1], the volume polynomial is Lorentzian whenever Y is irreducible and $\mathrm{H}_1,\ldots,\mathrm{H}_n$ are nef. We show that $\mathfrak{S}_w^\vee$ is a volume polynomial for suitable $Y = Y_w$ and H.

Let X be the product of projective spaces $(\mathbb{P}^{n-1})^n$. We write $x_{i1}, x_{i2},\ldots,x_{in}$ for the homogeneous coordinates of the i-th projective space, and write π_i for the i-th projection. We consider the map between the rank n vector bundles

$$\Psi : \bigoplus_{i=1}^n \mathcal{O}_X \longrightarrow \bigoplus_{j=1}^n \pi_j^*\mathcal{O}(1), \qquad \Psi(x) = (x_{ij})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant n}.$$

For $p,q \in [n]$, the induced map $\bigoplus_{i=1}^p \mathcal{O}_X \to \bigoplus_{j=1}^q \pi_j^*\mathcal{O}(1)$ will be denoted $\Psi_{p\times q}$. We set

$$Y = Y_w := \Big\{x \in X \mid \text{rank } \Psi_{p\times q}(x) \leqslant \text{rank } w_{p\times q} \text{ for all } p \text{ and } q\Big\},$$

where $w_{p\times q}$ is the $p\times q$ partial permutation matrix with ij-entry 1 for $w(i) = j$. The locus Y is defined by all minors of $(x_{ij})_{1\leqslant i\leqslant p, 1\leqslant j\leqslant q}$ of size one more than the rank of $w_{p\times q}$ for all p and q.

By [Ful92, Theorem 8.2], the fundamental class of Y in the Chow group of X is given by

$$[Y] = \mathfrak{S}_w(\mathrm{H}_1,\ldots,\mathrm{H}_n)\cap[X], \quad \mathrm{H}_i = c_1(\pi_i^*\mathcal{O}(1)).$$

An alternative proof of the displayed formula, in a more refined setting, was obtained in [KM05] through an explicit degeneration of Y. An important point for us is that Y is irreducible of expected codimension deg $\mathfrak{S}_w$ [Ful92]. For an elementary proof that the multi-homogeneous ideal defining Y is prime, see [MS05, Section 16.4]. The volume polynomial of Y with respect to $\mathrm{H} = (\mathrm{H}_1,\ldots,\mathrm{H}_n)$ is

$$\begin{aligned}\mathrm{vol}_{Y,\mathrm{H}}(x_1,\ldots,x_n) &= \frac{1}{\dim Y!}\int_Y (x_1\mathrm{H}_1+\cdots+x_n\mathrm{H}_n)^{\dim Y}\\ &= \frac{1}{\dim Y!}\int_X \mathfrak{S}_w(\mathrm{H}_1,\ldots,\mathrm{H}_n)(x_1\mathrm{H}_1+\cdots+x_n\mathrm{H}_n)^{\dim Y} = \mathfrak{S}_w^\vee(x_1,\ldots,x_n).\end{aligned}$$

The second equality is the projection formula, and the third equality follows from

$$\int_X \mathrm{H}^\mu = \begin{cases} 1 & \text{if } \mu = (n-1,\ldots,n-1), \\ 0 & \text{if } \mu \neq (n-1,\ldots,n-1). \end{cases}$$

Now the Lorentzian property of $\mathfrak{S}_w^\vee$ can be deduced from [BH19, Theorem 10.1]. □

Lemma 7. For any $\mu \in \mathbb{N}^n$ and any polynomial $f = f(x_1, \ldots, x_n)$,

$$\mathrm{N}(f) \text{ is Lorentzian if and only if } \mathrm{N}(x^\mu f) \text{ is Lorentzian.}$$

Proof. If a polynomial $g(x_1, \ldots, x_n)$ is Lorentzian, then so is its partial derivative

$$\partial^\mu g = \Big(\frac{\partial}{\partial x_1}\Big)^{\mu_1} \cdots \Big(\frac{\partial}{\partial x_n}\Big)^{\mu_n} g(x_1, \ldots, x_n).$$

Therefore, the "if" direction follows from the equality of linear operators

$$\partial^\mu \circ \mathrm{N} \circ x^\mu = \mathrm{N}.$$

The "only if" direction is a special case of [BH19, Corollary 6.8]. □

Proof of Theorem 3. As in the introduction, given a partition λ with m parts, we choose a large integer ℓ and write κ for the partition complementary to λ in the $m \times \ell$ rectangle. Choose another large integer n, and let w be the unique element of $\mathcal{S}_n$ satisfying

$$\kappa = \big(w(m) - m, \ldots, w(1) - 1\big) \;\text{ and }\; w(m) > w(m+1) < w(m+2) < \cdots < w(n).$$

The element w is the *Grassmannian permutation* in $\mathcal{S}_n$ with the *Lehmer code*

$$L(w) = (w(1) - 1, \ldots, w(m) - m, 0, \ldots, 0) = (\kappa_m, \ldots, \kappa_1, 0, \ldots, 0).$$

The Schubert polynomial of w satisfies

$$\mathfrak{S}_w(x_1, \ldots, x_n) = s_\kappa(x_1, \ldots, x_m) = x_1^\ell \cdots x_m^\ell s_\lambda(x_1^{-1}, \ldots, x_m^{-1}),$$

where the first equality is [Man01, Proposition 2.6.8] and the second equality is [FH91, Exercise 15.50]. By Theorem 6, we know that the polynomial $\mathfrak{S}_w^\vee$ is Lorentzian, which is equal to

$$\mathrm{N}(x_1^{n-1} \cdots x_n^{n-1} s_\kappa(x_1^{-1}, \ldots, x_m^{-1})) = \mathrm{N}(x^\mu s_\lambda(x_1, \ldots, x_m)) \text{ for some } \mu \in \mathbb{N}^n.$$

Therefore, by Lemma 7, the Lorentzian property of $\mathfrak{S}_w^\vee$ implies that of $\mathrm{N}(s_\lambda(x_1, \ldots, x_m))$. □

Proofs of Theorems 1 and 2. Since any nonzero Lorentzian polynomial is log-concave on the positive orthant, Theorem 1 follows from Theorem 3. For Theorem 2, we may suppose that

$$\mu_1 + \cdots + \mu_m = \lambda_1 + \cdots + \lambda_m \geqslant 2 \;\text{ and }\; \kappa := \mu - e_i - e_j \in \mathbb{N}^m.$$

We consider the quadratic form with at most one positive eigenvalue

$$\frac{\partial^{\kappa_1}}{\partial x_1^{\kappa_1}} \cdots \frac{\partial^{\kappa_m}}{\partial x_m^{\kappa_m}} \mathrm{N}(s_\lambda(x_1, \ldots, x_m)),$$

viewed as an $m \times m$ symmetric matrix. Its 2×2 principal submatrix corresponding to i and j is either identically zero or has exactly one positive eigenvalue, by Cauchy's interlacing theorem. The nonpositivity of the 2×2 principal minor gives the conclusion

$$K_{\lambda\mu}^2 \geqslant K_{\lambda\mu(i,j)} K_{\lambda\mu(j,i)}.$$

□

Proof of Corollary 4. The first part follows from Theorem 3 and [BH19, Corollary 6.8]. The second part follows from Theorem 3 and [BH19, Corollary 5.5]. □

In general, if h is a Lorentzian polynomial, then its normalization $\mathrm{N}(h)$ is a Lorentzian polynomial [BH19, Corollary 6.7]. We record here that Schur polynomials, before the normalization, need not be Lorentzian.

Example 8. The Schur polynomial of the partition $\lambda = (2, 0)$ in two variables is

$$s_\lambda(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2.$$

The quadratic form has eigenvalues $\frac{3}{2}$ and $\frac{1}{2}$, and hence s_λ is not Lorentzian.

A polynomial $f(x_1, \dots, x_m)$ is *stable* if f has no zeros in the product of m open upper half planes [Wag11]. Homogeneous stable polynomials with nonnegative coefficients are motivating examples of Lorentzian polynomials [BH19, Proposition 2.2]. We record here that normalized Schur polynomials, although Lorentzian, need not be stable.

Example 9. The normalized Schur polynomial of $\lambda = (3, 1, 1, 1, 1)$ in five variables is

$$\mathrm{N}(s_\lambda(x_1, \dots, x_5)) = \frac{1}{12}x_1x_2x_3x_4x_5\Big(\sum_{1 \leqslant i < j \leqslant 5} 3x_ix_j + \sum_{1 \leqslant i \leqslant 5} 2x_i^2\Big).$$

By [Wag11, Lemma 2.4], if $\mathrm{N}(s_\lambda)$ is stable, then so is its univariate specialization

$$\mathrm{N}(s_\lambda)|_{x_2=x_3=x_4=x_5=1} = \frac{1}{6}x_1\Big(x_1^2 + 6x_1 + 13\Big).$$

However, the displayed cubic has a pair of nonreal zeros, and hence $\mathrm{N}(s_\lambda)$ is not stable.

3. Ubiquity of Lorentzian polynomials

3.1. **Multiplicities of highest weight modules.** We point to [Hum08] for background on representation theory of semisimple Lie algebras. Let Λ be the integral weight lattice of the Lie algebra $\mathfrak{sl}_m(\mathbb{C})$, let $\varpi_1, \dots, \varpi_{m-1}$ be the fundamental weights, and let ρ be the sum of the fundamental weights. For $\lambda \in \Lambda$, we write $\mathrm{V}(\lambda)$ for the irreducible $\mathfrak{sl}_m(\mathbb{C})$-module with highest weight λ, and consider its decomposition into finite-dimensional weight spaces

$$\mathrm{V}(\lambda) = \bigoplus_\mu \mathrm{V}(\lambda)_\mu.$$

For $\mu \in \Lambda$ and distinct $i, j \in [m]$, we write $\mu(i,j)$ for the element $\mu + e_i - e_j \in \Lambda$.

Conjecture 10. For any $\lambda \in \Lambda$ and any $\mu \in \Lambda$, we have

$$(\dim \mathrm{V}(\lambda)_\mu)^2 \geqslant \dim \mathrm{V}(\lambda)_{\mu(i,j)} \dim \mathrm{V}(\lambda)_{\mu(j,i)} \quad \text{for any } i, j \in [m].$$

When λ is dominant, the dimension of the weight space $\mathrm{V}(\lambda)_\mu$ is the Kostka number $K_{\lambda\mu}$, and Theorem 2 shows that Conjecture 10 holds in this case. When λ is antidominant [Hum08, Section 4.4], $\mathrm{V}(\lambda)$ is the *Verma module* $\mathrm{M}(\lambda)$, the universal highest weight module of highest weight λ. We note that Conjecture 10 holds in this case as well.

Proposition 11. For any $\lambda \in \Lambda$ and any $\mu \in \Lambda$, we have

$$(\dim \mathrm{M}(\lambda)_\mu)^2 \geqslant \dim \mathrm{M}(\lambda)_{\mu(i,j)} \dim \mathrm{M}(\lambda)_{\mu(j,i)} \text{ for any } i, j \in [m].$$

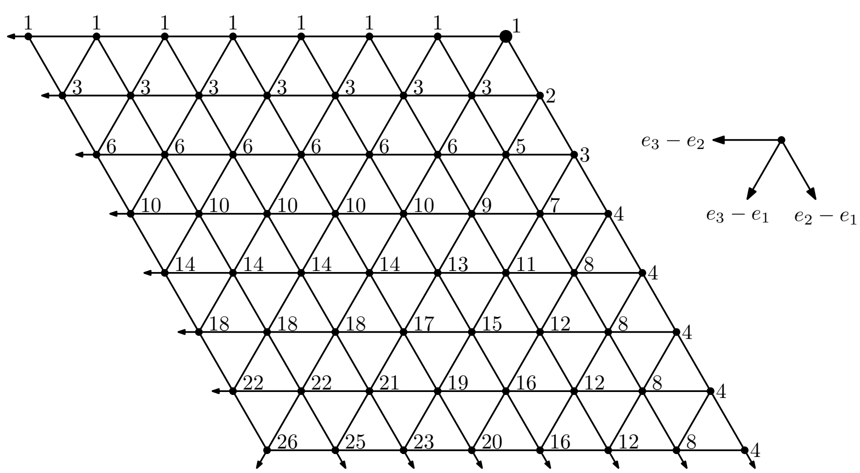
One may deduce Proposition 11 from its stronger variant Proposition 13 below.

Alternative proof. The Poincaré–Birkhoff–Witt theorem shows that the dimensions of the weight spaces are given by the Kostant partition function p:

$$\dim \mathrm{M}(\lambda)_\mu = p(\mu - \lambda) = \text{number of ways to write } \mu - \lambda \text{ as a sum of negative roots.}$$

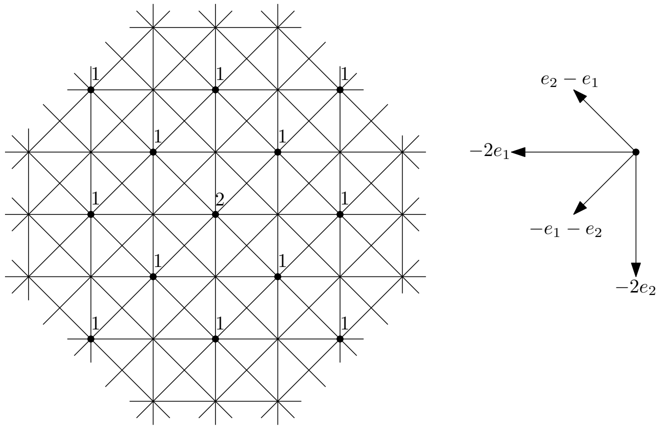
Lidskij's volume formula for flow polytopes shows that all Kostant partition function evaluations are mixed volumes of Minkowski sums of polytopes [BV08]. The Alexandrov–Fenchel inequality for mixed volumes [Sch14, Section 7.3] yields the desired log-concavity property. □

The diagram below shows some of the weight multiplicities of the irreducible $\mathfrak{sl}_4(\mathbb{C})$-module with highest weight $-2\varpi_1 - 3\varpi_2$. We start from the highlighted vertex $\varpi_1 - 6\varpi_2 - 3\varpi_3$ and walk along negative root directions in the hyperplane spanned by $e_2 - e_1$ and $e_3 - e_2$. In the shown region, the sequence of weight multiplicities along any line is log-concave, as predicted by Conjecture 10.



We note, however, that a naive analog of Conjecture 10 does not hold for symplectic Lie algebras. In the weight diagram of the irreducible representation of $\mathfrak{sp}_4(\mathbb{C})$ with highest weight $2\varpi_2$ shown below, the weight multiplicities along the two diagonals of the square do not form log-concave sequences.[9]

[9] Note that the Newton polytope of any homogeneous strongly log-concave polynomial is necessarily a generalized permutohedron of type A: Any edge of the Newton polytope should be parallel to $e_i - e_j$ for some i and j.



To strengthen Conjecture 10, we extend the normalization operator N to the space of Laurent generating functions by the formula

$$\mathrm{N}\left(\sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha\right) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \frac{x^\alpha}{\alpha!}.$$

For $\lambda \in \Lambda$, we introduce the Laurent generating functions

$$\mathrm{ch}_\lambda(x_1, \dots, x_m) = \sum_{\mu \in \Lambda} \dim \mathrm{V}(\lambda)_\mu \, x^{\mu - \lambda} \ \text{ and } \ \underline{\mathrm{ch}}_\lambda(x_1, \dots, x_m) = \sum_{\mu \in \Lambda} \dim \mathrm{M}(\lambda)_\mu \, x^{\mu - \lambda}.$$

Note that every monomial appearing in the shifted characters ch_λ and $\underline{\mathrm{ch}}_\lambda$ is a product of degree zero monomials of the form $x_i x_j^{-1}$ with $i > j$.

We tested the following statement for $\lambda = -w\rho - \rho$ and $\delta = (1, \dots, 1)$, for all permutations w in $\mathcal{S}_m$ for $m \leqslant 6$.[10]

Conjecture 12. The polynomial $\mathrm{N}(x^\delta \, \mathrm{ch}_\lambda(x_1, \dots, x_m))$ is Lorentzian for any $\lambda \in \Lambda$ and $\delta \in \mathbb{N}^m$.

For example, when $m = 4$ and $\lambda = -w\rho - \rho$ for the transposition $w = (1, 2)$, we have

$$\begin{aligned}\mathrm{N}(x_1x_2x_3x_4\mathrm{ch}_\lambda(x_1, x_2, x_3, x_4)) &= \frac{4}{24}x_4^4 + \frac{2}{6}x_1x_4^3 + \frac{2}{6}x_2x_4^3 + \frac{4}{6}x_3x_4^3 + \frac{3}{4}x_3^2x_4^2 \\ &+ \frac{1}{2}x_1x_2x_4^2 + \frac{2}{2}x_1x_3x_4^2 + \frac{2}{2}x_2x_3x_4^2 + \frac{1}{6}x_3^3x_4 + \frac{1}{2}x_1x_3^2x_4 + \frac{1}{2}x_2x_3^2x_4 + \frac{1}{1}x_1x_2x_3x_4,\end{aligned}$$

which is a Lorentzian polynomial. In general, the homogeneous polynomial $\mathrm{N}(x^\delta \mathrm{ch}_\lambda)$ can be computed using the Kazhdan–Lusztig theory [Hum08, Chapter 8].

Theorem 3 and Lemma 7 show that Conjecture 12 holds for any δ when λ is dominant. We show that Conjecture 12 holds for any δ when λ is antidominant.

Proposition 13. The polynomial $\mathrm{N}(x^\delta \, \underline{\mathrm{ch}}_\lambda(x_1, \dots, x_m))$ is Lorentzian for any $\lambda \in \Lambda$ and $\delta \in \mathbb{N}^m$.

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[10]We point to `https://github.com/avstdi/Lorentzian-Polynomials` for code supporting the computations in Section 3.

Proof. Recall that the dimensions of the weight spaces of $\mathrm{M}(\lambda)$ are given by the Kostant partition function p. In other words, we have

$$\underline{\mathrm{ch}}_\lambda(x_1,\ldots,x_m)=\prod_{i>j}(1+x_ix_j^{-1}+x_i^2x_j^{-2}+\cdots).$$

Note that, in the expansion of the above product,[11] only the terms of degree at least $-\delta$ contribute to $\mathrm{N}(x^\delta\underline{\mathrm{ch}}_\lambda)$. Therefore, we may choose a suitably large $\alpha\in\mathbb{N}^m$ depending on $\delta\in\mathbb{N}^m$ so that

$$\mathrm{N}(x^\delta\underline{\mathrm{ch}}_\lambda)=\mathrm{N}(x^\delta x^{-\beta}\prod_{i>j}(x_j^{\alpha_j}+x_ix_j^{\alpha_j-1}+\cdots+x_i^{\alpha_j})),\quad \text{where } \beta_i=(m-i)\alpha_i \text{ for all } i.$$

Observe that the right-hand side is the β-th partial derivative of the normalized product of x^δ and $\sum_k x_i^{\alpha_j-k}x_j^k$, whose normalization is the Lorentzian polynomial

$$\mathrm{N}(x_j^{\alpha_j}+x_ix_j^{\alpha_j-1}+\cdots+x_i^{\alpha_j})=\frac{1}{\alpha_j!}(x_i+x_j)^{\alpha_j}.$$

The conclusion now follows from [BH19, Corollary 6.8]. □

Conjecture 10 for λ and μ follows from Conjecture 12 for λ and a sufficiently large δ. Conjecture 12 for λ and δ follows from Conjecture 12 for λ and any δ' larger than δ componentwise.

3.2. **Schubert polynomials.** For $w\in\mathcal{S}_n$ and $\mu\in\mathbb{Z}^n$, we define the number $K_{w\mu}$ by

$$\mathfrak{S}_w(x_1,\ldots,x_n)=\sum_\mu K_{w\mu}x^\mu.$$

As before, for $\mu\in\mathbb{Z}^n$ and distinct $i,j\in[m]$, we set

$$\mu(i,j)=\mu+e_i-e_j.$$

We note that Theorem 2 can be strengthened as follows.

Proposition 14. For any $w\in\mathcal{S}_n$ and any $\mu\in\mathbb{N}^n$, we have

$$K_{w\mu}^2\geqslant K_{w\mu(i,j)}K_{w\mu(j,i)} \text{ for any } i,j\in[n].$$

Proof. By Theorem 6, the polynomial $\mathfrak{S}_w^\vee$ is Lorentzian. The inequality follows from [BH19, Proposition 9.4] applied to the Lorentzian polynomial $\mathfrak{S}_w^\vee$. □

Are normalized Schubert polynomials Lorentzian? We tested the following statement for all permutations in $\mathcal{S}_n$ for $n\leqslant 8$.

Conjecture 15. The polynomial $\mathrm{N}(\mathfrak{S}_w(x_1,\ldots,x_n))$ is Lorentzian for any $w\in\mathcal{S}_n$.

[11] It is clear that the product is well-defined. Officially, the product occurs in the ring of formal characters of the category $\mathcal{O}$ of $\mathfrak{sl}_m(\mathbb{C})$-modules, denoted $\mathcal{X}$ in [Hum08, Section 1.15].

More generally, we conjecture that, for double Schubert polynomials [MS05, Section 15.5],

$$\mathrm{N}(\mathfrak{S}_w(x_1, \dots, x_n, -y_1, \dots, -y_n)) \text{ is Lorentzian for any } w \in \mathcal{S}_n.$$

This would imply that the support of any double Schubert polynomial is M-convex, and hence "saturated" [MTY17, Conjecture 5.2].

Proposition 16. The support of $\mathfrak{S}_w(x_1, \dots, x_n)$ is M-convex for any $w \in \mathcal{S}_n$.

Proposition 16 was conjectured in [MTY17, Conjecture 5.1] and proved in [FMS18] using an explicit description of flagged Schur modules. Here we give an alternative proof based on Theorem 6. A similar argument can be used more generally to show that the supports of single quiver polynomials appearing in [MS05, Section 17.4] are M-convex.

Proof. By Theorem 6, the support of $\mathfrak{S}_w^\vee$ is M-convex. It is straightforward to check using the definition of M-convexity the general fact that, if the support of $h(x_1, \dots, x_n)$ is M-convex, then the support of $x^\mu h(x_1^{-1}, \dots, x_n^{-1})$ is M-convex for any monomial x^μ divisible by all monomials in the support of h.[12] □

Proposition 17. Conjecture 15 holds when $w \in \mathcal{S}_n$ avoids the patterns 1423 and 1432.

Sketch of Proof. By [BH19, Corollary 6.7], the Lorentzian property of $\mathfrak{S}_w$ implies that of $\mathrm{N}(\mathfrak{S}_w)$. We deduce the Lorentzian property of $\mathfrak{S}_w$ from known results on Schubert and Lorentzian polynomials, for permutations avoiding 1423 and 1432.

It is shown in [FMS18, Theorem 7] that, for any $w \in \mathcal{S}_n$, the support of $\mathfrak{S}_w$ is the set of integral points in the Minkowski sum of n matroid polytopes. The set J_w of integral points in the Cartesian product of these matroid polytopes is an M-convex subset of $\mathbb{N}^{n \times n}$, and hence the generating function f_w of J_w is a Lorentzian polynomial in n^2 variables x_{ij} [BH19, Theorem 7.1]. Since any nonnegative linear change of coordinates preserves the Lorentzian property [BH19, Theorem 2.10], substituting the variables x_{ij} by x_i in the generating function f_w gives a Lorentzian polynomial. According to [FMS19, Corollary 5.6] and [FG19, Theorem 1.1], this specialization of f_w coincides with $\mathfrak{S}_w$ when w avoids 1423 and 1432, and thus $\mathfrak{S}_w$ is Lorentzian for such permutations. □

We note that the Schubert polynomials $\mathfrak{S}_{1423}$ and $\mathfrak{S}_{1432}$ are not Lorentzian.

3.3. **Degree polynomials.** Let $w < w(i,j)$ be a covering relation in the Bruhat order of $\mathcal{S}_n$ labelled by the transposition of $i < j$ in $[n]$. The *Chevalley multiplicity* is the assignment

$$w < w(i,j) \longmapsto \sum_{i \leqslant k < j} x_k,$$

[12] The general fact extends matroid duality [Oxl11, Chapter 2], which is the special case $\mu = (1, \dots, 1)$.

where x_k are independent variables. The *degree polynomial* of $w \in \mathcal{S}_n$ is the generating function

$$\mathfrak{D}_w(x_1, \dots, x_{n-1}) = \sum_{\mathrm{C}} m_{\mathrm{C}}(x_1, \dots, x_{n-1}),$$

where the sum is over all saturated chains C from the identity permutation to w, and m_{C} is the product of Chevalley multiplicities of the covering relations in C. The degree polynomials were introduced by Bernstein, Gelfand, and Gelfand [BGG73] and studied from a combinatorial perspective by Postnikov and Stanley [PS09].

Proposition 18. The degree polynomial $\mathfrak{D}_w(x_1, \dots, x_{n-1})$ is Lorentzian for any $w \in \mathcal{S}_n$.

Proof. Let B be the group of upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$, and let X_w be the closure of the B-orbit of the permutation matrix corresponding to w in the flag variety $\mathrm{GL}_n(\mathbb{C})/B$. By [PS09, Proposition 4.2], the degree polynomial of w is, up to a normalizing constant, the volume polynomial of X_w with respect to the line bundles associated to the fundamental weights $\varpi_1, \dots, \varpi_{n-1}$. The conclusion follows from [BH19, Theorem 10.1]. $\square$

The same argument shows that the analogous statement holds for Weyl groups in other types.

3.4. **Skew Schur polynomials.** Let λ/ν be a skew Young diagram. The *skew Schur polynomial* of λ/ν in m variables is the generating function

$$s_{\lambda/\nu}(x_1, \dots, x_m) = \sum_{\mathrm{T}} x^{\mu(\mathrm{T})}, \quad x^{\mu(\mathrm{T})} = x_1^{\mu_1(\mathrm{T})} \cdots x_m^{\mu_m(\mathrm{T})},$$

where the sum is over all Young tableaux T of skew shape λ/ν with entries from $[m]$, and

$$\mu_i(\mathrm{T}) = \text{the number of } i\text{'s among the entries of } \mathrm{T}, \quad \text{for } i = 1, \dots, m.$$

Are normalized skew Schur polynomials Lorentzian? We tested the following statement for all partitions λ with at most 12 boxes and at most 6 parts.

Conjecture 19. The polynomial $\mathrm{N}(s_{\lambda/\nu}(x_1, \dots, x_m))$ is Lorentzian for any λ/ν.

Theorem 3 shows that Conjecture 19 holds when ν is zero, and Corollary 4 provides some further evidence. We remark that the M-convexity of the support of any skew Schur polynomial can be deduced from [MTY17, Proposition 2.9].

3.5. **Schur P-polynomials.** Let λ be a *strict partition*, that is, a decreasing sequence of positive integers. The *Schur P-polynomial* of λ in m variables is the generating function

$$P_\lambda(x_1, \dots, x_m) = \sum_{\mathrm{T}} x^{\mu(\mathrm{T})}, \quad x^{\mu(\mathrm{T})} = x_1^{\mu_1(\mathrm{T})} \cdots x_m^{\mu_m(\mathrm{T})},$$

where the sum is over all marked shifted Young tableaux of shape λ with entries from $[m]$. See [Mac15, Chapter III] for this and other equivalent definitions of the polynomial P_λ.

Are normalized Schur P-polynomials Lorentzian? We tested the following statement for all strict partitions λ with $\lambda_1 \leqslant 12$ and at most 4 parts.

Conjecture 20. The polynomial $\mathrm{N}(P_\lambda(x_1, \ldots, x_m))$ is Lorentzian for any strict partition λ.

The M-convexity of the support of P_λ was observed in [MTY17, Proposition 3.5].

3.6. **Grothendieck polynomials.** *Grothendieck polynomials* are polynomial representatives of the Schubert classes in the Grothendieck ring introduced by Lascoux and Schützenberger [LS83]. If w is the longest permutation $w_\circ \in \mathcal{S}_n$, then the Grothendieck polynomial of w is the monomial

$$\mathfrak{G}_{w_\circ}(x_1, \ldots, x_n) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1.$$

In general, if $w(i) > w(i+1)$ for some i and s_i is the adjacent transposition $(i\ i+1)$, then

$$\mathfrak{G}_{ws_i}(x_1, \ldots, x_n) = \pi_i \mathfrak{G}_w(x_1, \ldots, x_n), \ \text{ where } \pi_i = \partial_i - \partial_i x_{i+1}.$$

Let $\ell(w)$ be the degree of the Schubert polynomial of w, let $d(w)$ be the degree of the Grothendieck polynomial of w, and let $\mathfrak{G}_w^k$ be the degree $\ell(w)+k$ homogeneous component of the Grothendieck polynomial.

Conjecture 21. The polynomial $(-1)^k \mathrm{N}(\mathfrak{G}_w^k(x_1, \ldots, x_n))$ is Lorentzian for any $w \in \mathcal{S}_n$ and $k \in \mathbb{N}$.

The M-convexity of the support of $\mathfrak{G}_w^k$ was conjectured in [MS17, Conjecture 5.1] and proved in [EY17] when w is a Grassmannian permutation. Conjecture 21 implies Conjecture 15 because the degree $\ell(w)$ homogeneous component of $\mathfrak{G}_w$ is the Schubert polynomial $\mathfrak{S}_w$.

We may strengthen Conjecture 21 in terms of the *homogeneous Grothendieck polynomial*

$$\widetilde{\mathfrak{G}}_w(x_1, \ldots, x_n, z) := \sum_{k=0}^{d(w)-\ell(w)} (-1)^k \mathfrak{G}_w^k(x_1, \ldots, x_n) z^{d(w)-\ell(w)-k},$$

where z is a new variable. Are normalized homogeneous Grothendieck polynomials Lorentzian? We tested the following statement for all permutations in $\mathcal{S}_n$ for $n \leqslant 7$.

Conjecture 22. The polynomial $\mathrm{N}(\widetilde{\mathfrak{G}}_w(x_1, \ldots, x_n, z))$ is Lorentzian for any $w \in \mathcal{S}_n$.

Conjecture 22 implies Conjecture 21 because taking partial derivatives and setting a variable equal to zero preserve the Lorentzian property. We expect an analogous Lorentzian property for double Grothendieck polynomials.

3.7. **Key polynomials.** *Key polynomials* were introduced by Demazure for Weyl groups [Dem74] and studied by Lascoux and Schützenberger for symmetric groups [LS90]. When $\mu \in \mathbb{N}^n$ is a partition, the key polynomial of μ is the monomial

$$\kappa_\mu(x_1, \ldots, x_n) = x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}.$$

If $\mu_i < \mu_{i+1}$ for some i and s_i is the adjacent transposition $(i\ i+1)$, then

$$\kappa_\mu(x_1, \ldots, x_n) = \partial_i x_i \kappa_\nu, \ \text{ where } \nu = \mu s_i = (\mu_1, \ldots, \mu_{i+1}, \mu_i, \ldots, \mu_n).$$

We refer to [RS95] for more information about key polynomials.

Are normalized key polynomials Lorentzian? We tested the following statement for all compositions μ with at most 12 boxes and at most 6 parts.

Conjecture 23. The polynomial $\mathrm{N}(\kappa_\mu(x_1,\dots,x_n))$ is Lorentzian for any $\mu \in \mathbb{N}^n$.

Theorem 3 shows that Conjecture 23 holds when μ is a weakly increasing sequence of nonnegative integers, because in this case the key polynomial of μ is a Schur polynomial. The M-convexity of the supports of key polynomials was conjectured in [MTY17, Conjecture 3.13] and proved in [FMS18].

We remark that key polynomials [Dem74] and Schubert polynomials [KP87] are both characters of *flagged Schur modules*.[13] It is shown in [FMS18, Theorem 11] that the character of any flagged Schur module has M-convex support. Are normalized characters of flagged Schur modules Lorentzian?

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