# Logarithmic concavity of weight multiplicities for irreducible $\mathfrak{sl}_n(\mathbb{C})$-representations

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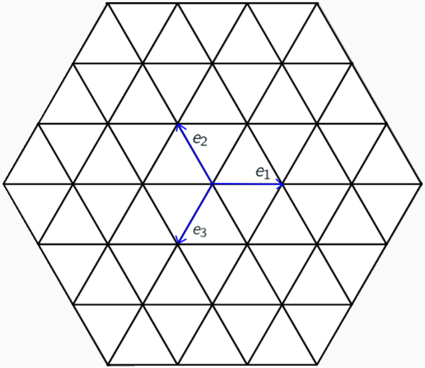
# Motivation from representation theory

# Weight lattices

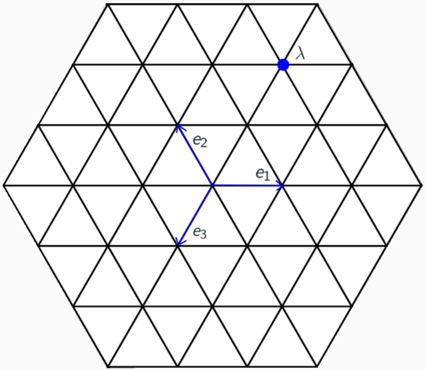
Integral weight lattice of $\mathfrak{sl}_n(\mathbb{C})$:

$$\Lambda := \mathbb{Z}\{e_1, \ldots, e_n\} / \left( \sum_{i=1}^{n} e_i = 0 \right)$$



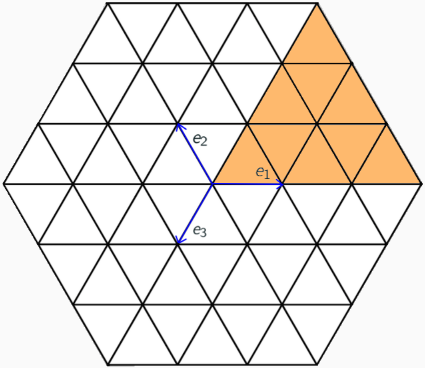
# Irreducible representations

$$\begin{array}{ccc} \Lambda & \longrightarrow & \{\text{irreducible representations of } \mathfrak{sl}_n(\mathbb{C})\} \\ \lambda & \longmapsto & V(\lambda) \end{array}$$



# Dominant Weyl chamber

$V(\lambda)$ is finite dimensional if and only if $\lambda$ is dominant.



# Weight multiplicities

Each $V(\lambda)$ has a weight space decomposition

$$V(\lambda) = \bigoplus_{\mu} V(\lambda)_{\mu}.$$

All $V(\lambda)_{\mu}$ are **finite dimensional**.

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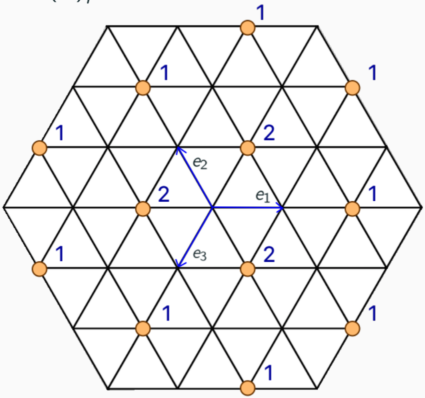


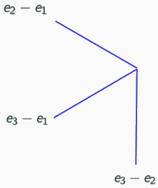

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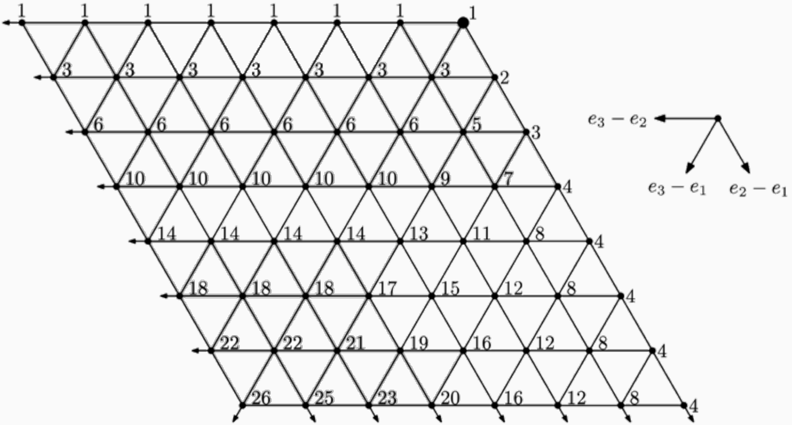

# Log-concavity of weight multiplicities

**Theorem (Huh–M.–Mészáros–St. Dizier 2019)**

*For $\lambda, \mu \in \Lambda$ with $\lambda$ dominant, we have*

$$(\dim V(\lambda)_\mu)^2 \geq \dim V(\lambda)_{\mu+e_i-e_j} \dim V(\lambda)_{\mu-e_i+e_j}$$

*for any $i, j \in [n]$.*

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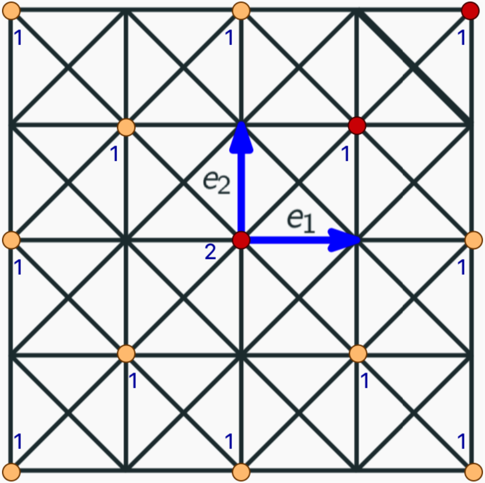
It's easy for $\mathfrak{sl}_2(\mathbb{C})$ because all weight spaces are one dimensional.

# Counterexample in other types

The theorem fails for $\mathfrak{sp}_4(\mathbb{C})$!



# Antidominant Weyl chamber

If $\lambda \in \Lambda$ is antidominant, then $V(\lambda) = M(\lambda)$. $\leftarrow$ Verma module

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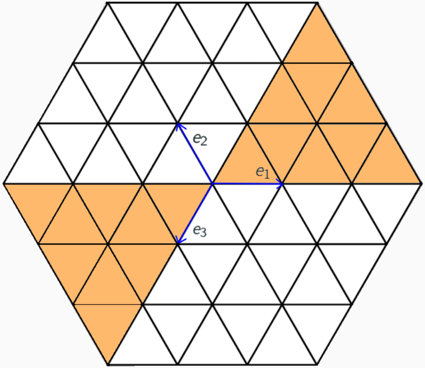
**Proof idea.**

It is known that

$$\dim M(\lambda)_\mu = p(\mu - \lambda), \leftarrow \text{Kostant's partition function}$$

which is the number of ways of writing $\mu - \lambda$ as a sum of negative roots. □

# Main conjecture



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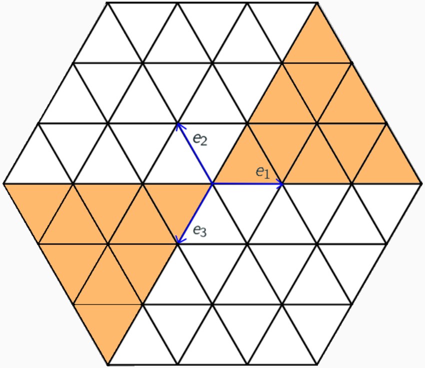


**Conjecture (Huh–M.–Mészáros–St. Dizier 2019)**

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# Schur polynomials

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**Definition**

The Schur polynomial (in $n$ variables) of a partition $\lambda$ is

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \mathrm{SSYT}} x^{\mu(T)}, \qquad x^{\mu(T)} := x_1^{\mu_1(T)} \cdots x_2^{\mu_2(T)}.$$

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For $\lambda = (2,1)$, we have

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

So,

$$s_{(2,1)}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2.$$

# Continuous theorem

Grouping terms with the same $\mu$ gives

$$s_\lambda(x_1, \ldots, x_n) = \sum_\mu K_{\lambda\mu} x^\mu. \qquad \leftarrow K_{\lambda\mu}, \text{ Kostka number}$$

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**Continuous Theorem (Huh–M.–Mészáros–St. Dizier 2019)**

*For any partition $\lambda$, we have*

$$N(s_\lambda(x_1, \ldots, x_n)) = \sum_\mu K_{\lambda\mu} \frac{x^\mu}{\mu!}$$

*is either identically* $0$ *or* $\log(N(s_\lambda))$ *is a concave function on* $\mathbb{R}^n_{>0}$.

# Discrete theorem

**Discrete Theorem (Huh–M.–Mészáros–St. Dizier 2019)**

*For any partition* $\lambda$ *and* $\mu \in \mathbb{N}^n$, *we have*

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The Discrete Theorem implies our first theorem on weight multiplicities because

$$\dim V(\lambda)_\mu = K_{\lambda\mu}.$$

# Okounkov's Conjecture

Littlewood–Richardson coefficients $c_{\lambda\kappa}^{\nu}$ are given by

$$V(\lambda) \otimes V(\kappa) \simeq \bigoplus_{\nu} V(\nu)^{c_{\lambda\kappa}^{\nu}}.$$

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Counterexample due to Chindris–Derksen–Weyman in 2007.

# Special case of Okounkov's Conjecture

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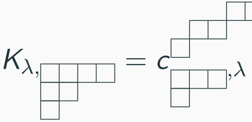

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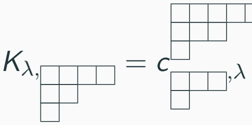

# Main theorem

**Main Theorem (Huh–M.–Mészáros–St. Dizier 2019)**

*For any partition $\lambda$, the normalized Schur polynomial*

$$N(s_\lambda(x_1, \ldots, x_n))$$

*is Lorentzian.*

# Lorentzian polynomials

# Lorentzian polynomials

**Definition (Brändén–Huh 2019)**

A degree $d$ homogeneous polynomial $h(x_1, \ldots, x_n)$ is Lorentzian if

- all coefficients of $h$ are nonnegative,
- $\mathrm{supp}(h)$ has the *exchange property*, and
- the quadratic form $\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_{d-2}}}(h)$ has at most one positive eigenvalue for all $i_1, \ldots, i_{d-2} \in [n]$.

# Examples of Lorentzian polynomials

**Nonexample:**

$s_{(2,0)}(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$

Its matrix is $\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$.

Eigenvalues are 3/2 and 1/2.

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**Example:**

$N(s_{(2,0)}(x_1, x_2)) = \frac{x_1^2}{2} + x_1x_2 + \frac{x_2^2}{2}$

Its matrix is $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

Eigenvalues are 0 and 1.

# Consequences of the Lorentzian property

**Theorem (Brändén–Huh 2019)**

*If $f = \sum_\alpha \frac{c_\alpha}{\alpha!} x^\alpha$ is a Lorentzian polynomial, then*

- *$f$ is either identically $0$ or $\log(f)$ is concave on $\mathbb{R}^n_{>0}$, and*
- *$c_\alpha^2 \geq c_{\alpha+e_i-e_j} c_{\alpha-e_i+e_j}$ for all $\alpha$ and for all $i, j \in [n]$.*

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This implies our Continuous Theorem and our Discrete Theorem.

# Some words about the proof

Naïve attempt: induction

**Example:**

$N(s_{(2,1)}(x_1, x_2)) = \frac{x_1^2 x_2}{2} + \frac{x_1 x_2^2}{2}$

$\frac{\partial}{\partial x_1} N(s_{(2,1)}(x_1, x_2)) = x_1 x_2 + \frac{x_2^2}{2}$ ← not symmetric!

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$\frac{\partial}{\partial x_1} N(s_{(2,1)}(x_1, x_2)) = x_1 x_2 + \frac{x_2^2}{2} \leftarrow$ not symmetric!

Instead:

We show $N(s_\lambda(x_1, \ldots, x_n))$ is a volume polynomial.

# Conjectural Lorentzian polynomials

| Polynomial | Tested for |
|---|---|
| Schubert: $N(\mathfrak{S}_w(x_1, \ldots, x_n))$ | $n \leq 8$ |
| Skew Schur: $N(s_{\lambda/\mu}(x_1, \ldots, x_n))$ | $\lambda$ with $\leq 12$ boxes and $\leq 6$ parts |
| Schur P: $N(P_\lambda(x_1, \ldots, x_n))$ | strict $\lambda$ with $\lambda_1 \leq 12$ and $\leq 4$ parts |
| homog. Grothendieck: $N(\widetilde{\mathfrak{G}}_w(x_1, \ldots, x_n, z))$ | $n \leq 7$ |
| Key: $N(\kappa_\mu(x_1, \ldots, x_n))$ | compositions $\mu$ with $\leq 12$ boxes and $\leq 6$ parts |

https://github.com/avstdi/Lorentzian-Polynomials

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