POINCARÉ POLYNOMIALS ASSOCIATED TO GEOMETRIC LATTICES

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A number of graded objects associated to a matroid have played a recent role in the resolution of long-standing conjectures in the field of matroid theory. The goal of this report is to survey what is known about their Poincaré polynomials.

To a (loopless) matroid M, one may associate the following graded objects:

(1) the intersection cohomology module $\mathrm{IH}(\mathsf{M})$ [BHM+20],
(2) its "stalk at the empty flat" $\mathrm{IH}(\mathsf{M})_\varnothing$ [BHM+20],
(3) the augmented Chow ring $\mathrm{CH}(\mathsf{M})$ [BHM+22], and
(4) the Chow ring $\underline{\mathrm{CH}}(\mathsf{M})$ [FY04].

Each of these objects has a topological interpretation when the matroid M is realizable by a collection of vectors in a complex vector space V. These interpretations hinge on a certain singular projective variety $Y_\mathcal{A}$, introduced in [AB16] and now called the matroid Schubert variety, that is constructed from the vector space V. It gets its name from the analogous role it plays in the Kazhdan–Lusztig theory of matroids [EPW16] that the classical Schubert varieties play in the Kazhdan–Lusztig theory of Coxeter groups [KL79, KL80].

The matroid Schubert variety $Y_\mathcal{A}$ has a canonical resolution of singularities $\pi_\mathcal{A}\colon \widetilde{Y}_\mathcal{A} \to Y_\mathcal{A}$, where $\widetilde{Y}_\mathcal{A}$ is the so-called augmented wonderful variety. In the realizable case, the respective graded objects in the numbered list above are isomorphic (with a degree-doubling isomorphism) to the following topological objects:

(1) the intersection cohomology $\mathrm{IH}(Y_\mathcal{A})$ of $Y_\mathcal{A}$,
(2) the local intersection cohomology $\mathrm{IH}_{(\infty,\ldots,\infty)}(Y_\mathcal{A})$ of $Y_\mathcal{A}$ at the point $(\infty,\ldots,\infty) \in Y_\mathcal{A}$,
(3) the cohomology $\mathrm{H}(\widetilde{Y}_\mathcal{A})$ of $\widetilde{Y}_\mathcal{A}$, and
(4) the cohomology $\mathrm{H}(\pi_\mathcal{A}^{-1}(\infty,\ldots,\infty))$ of the fiber $\pi_\mathcal{A}^{-1}(\infty,\ldots,\infty)$.

Although most matroids are not realizable [Nel18], the miracle is that arbitrary matroids *behave as if they were geometric objects*. Indeed, $\mathrm{IH}(\mathsf{M})$, $\mathrm{CH}(\mathsf{M})$, and $\underline{\mathrm{CH}}(\mathsf{M})$ satisfy the Kähler package [BHM+20, AHK18, BHM+22], a trio of important results consisting of Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations. The Heron–Rota–Welsh conjecture on the log-concavity of the characteristic polynomial of M [Her72, Rot71, Wel76] follows from the Hodge–Riemann relations for $\underline{\mathrm{CH}}(\mathsf{M})$, and both the Dowling–Wilson top-heavy conjecture on the shape of the lattice of flats of M [DW74, DW75] and the nonnegativity of the Kazhdan–Lusztig and Z-polynomials of a matroid [EPW16, GPY17] (see the bulleted list below) follow from the hard Lefschetz theorem for $\mathrm{IH}(\mathsf{M})$. There has been an industry of recent interest in studying the respective Poincaré polynomials of the graded objects in the first bulleted list:

(1) the Z-polynomial $Z_{\mathsf{M}}(t)$ of a matroid M [PXY18],
(2) the Kazhdan–Lusztig polynomial $P_{\mathsf{M}}(t)$ of a matroid M [EPW16],
(3) the augmented Chow polynomial $\mathrm{H}_{\mathsf{M}}(t)$ of a matroid M [FMSV24], and
(4) the Chow polynomial $\underline{\mathrm{H}}_{\mathsf{M}}(t)$ of a matroid M [FMSV24].

Conjecture 0.1 ([PXY18, GPY17, Ste21, FS22])**.** The polynomials $Z_{\mathsf{M}}(t)$, $P_{\mathsf{M}}(t)$, $\mathrm{H}_{\mathsf{M}}(t)$, and $\underline{\mathrm{H}}_{\mathsf{M}}(t)$ are real-rooted for every matroid M.

Real-rootedness is the strictest condition in a sequence of implications involving interesting combinatorial patterns for single-variable polynomials whose coefficient sequence consists of nonnegative integers and has no internal zeros:

$$\begin{array}{ccccc} & \nearrow & \gamma\text{-positivity}^{1} & \searrow & \\ \text{real-rootedness} & \Longrightarrow & \text{log-concavity} & \Longrightarrow & \text{unimodality} \end{array}$$

The polynomials $P_{\mathsf{M}}(t)$ and $Z_{\mathsf{M}}(t)$ are real-rooted in the following cases: when $\mathsf{M} = \mathsf{U}_{d,n}$ is uniform of rank d on n elements for all $d \geqslant 1$ and all $2 \leqslant n - d \leqslant 15$ [GLX+21]; when M is a fan, wheel, or whirl matroid [LXY22]; and when M is a sparse paving matroid with at most 30 elements [FV22]. Log-concavity of $P_{\mathsf{M}}(t)$ holds for all uniform matroids [XZ23].

Poincaré duality and the hard Lefschetz theorem for $\underline{\mathrm{CH}}(\mathsf{M})$ and $\mathrm{CH}(\mathsf{M})$ imply unimodality for $\underline{\mathrm{H}}_{\mathsf{M}}(t)$ and $\mathrm{H}_{\mathsf{M}}(t)$, and the same theorems for $\mathrm{IH}(\mathsf{M})$ imply unimodality for $Z_{\mathsf{M}}(t)$. In [FMSV24], the semi-small decomposition of $\underline{\mathrm{CH}}(\mathsf{M})$ and $\mathrm{CH}(\mathsf{M})$ from [BHM+22] are used to prove the γ-positivity of $\underline{\mathrm{H}}_{\mathsf{M}}(t)$ and $\mathrm{H}_{\mathsf{M}}(t)$; and, a result of Braden–Vysogorets [BV20] is used to prove γ-positivity for $Z_{\mathsf{M}}(t)$.[2]

Whereas $P_{\mathsf{M}}(t)$ and $Z_{\mathsf{M}}(t)$ were defined recursively in [EPW16, PXY18], and their interpretation as Poincaré polynomials was established later [BHM+20], the story for $\mathrm{H}_{\mathsf{M}}(t)$ and $\underline{\mathrm{H}}_{\mathsf{M}}(t)$ is the reverse. In [FMSV24] a recursive formula was given for the Poincaré polynomials $\mathrm{H}_{\mathsf{M}}(t)$ and $\underline{\mathrm{H}}_{\mathsf{M}}(t)$, paralleling the definition of $P_{\mathsf{M}}(t)$ and $Z_{\mathsf{M}}(t)$. This formula leads to several consequences [FMSV24]: $\mathrm{H}_{\mathsf{M}}(t)$ and $\underline{\mathrm{H}}_{\mathsf{M}}(t)$ are real-rooted for M sparse paving with at most 40 elements; $\mathrm{H}_{\mathsf{M}}(t)$

[1] The notion of γ-positivity is only meaningful for palindromic polynomials, so we do not consider it for $P_{\mathsf{M}}(t)$.

[2] The former was observed independently by Botong Wang, and the latter resolved a conjecture in [FNV23].

is real-rooted for M uniform since it is an example of a generalized binomial Eulerian polynomial of Haglund–Zhang [HZ19]; and $\mathrm{H}_{\mathsf{M}}(t)$ (respectively $\underline{\mathrm{H}}_{\mathsf{M}}(t)$) is real-rooted for all M with rank less than five (respectively six) by using the fact that $\underline{\mathrm{CH}}(\mathsf{M})$ and $\mathrm{CH}(\mathsf{M})$ are Koszul algebras [MM23] together with results from [RW05].

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