SYZYGIES OF POLYMATROIDAL IDEALS

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ABSTRACT. We introduce the cave polynomial of a polymatroid and show that it yields a valuative function on polymatroids. The support of this polynomial after homogenization is again a polymatroid. The cave polynomial gives a K-theoretic description of a polymatroid in the augmented K-ring of a multisymmetric lift. As applications, we settle two conjectures: one by Bandari, Bayati, and Herzog regarding polymatroidal ideals, and another by Castillo, Cid-Ruiz, Mohammadi, and Montaño regarding the Möbius support of a polymatroid.

1. INTRODUCTION

A *polymatroid* $\mathscr{P}$ on the set $[p] = \{1,\ldots,p\}$ with cage $\mathbf{m} = (m_1,\ldots,m_p) \in \mathbb{N}^p$ is given by a function $\mathrm{rk}_{\mathscr{P}}\colon 2^{[p]} \to \mathbb{N}$ satisfying the following properties:

(i) (Normalization) $\mathrm{rk}_{\mathscr{P}}(\varnothing) = 0$.
(ii) (Monotonicity) $\mathrm{rk}_{\mathscr{P}}(J_1) \leqslant \mathrm{rk}_{\mathscr{P}}(J_2)$ if $J_1 \subseteq J_2 \subseteq [p]$.
(iii) (Submodularity) $\mathrm{rk}_{\mathscr{P}}(J_1 \cap J_2) + \mathrm{rk}_{\mathscr{P}}(J_1 \cup J_2) \leqslant \mathrm{rk}_{\mathscr{P}}(J_1) + \mathrm{rk}_{\mathscr{P}}(J_2)$ for all $J_1, J_2 \subseteq [p]$.
(iv) (Cage) $\mathrm{rk}_{\mathscr{P}}(\{i\}) \leqslant m_i$ for all $i \in [p]$.

We say that $\mathrm{rk}_{\mathscr{P}}\colon 2^{[p]} \to \mathbb{N}$ is the *rank function* of $\mathscr{P}$ and that the *rank* of $\mathscr{P}$ is given by $\mathrm{rk}(\mathscr{P}) = \mathrm{rk}_{\mathscr{P}}([p])$. A polymatroid with cage $\mathbf{m} = (1,\ldots,1)$ is called a *matroid*.

Let $R = \Bbbk[x_1,\ldots,x_p]$ be a polynomial ring over a field $\Bbbk$. Let $\mathscr{P}$ be a polymatroid on the set $[p]$ with cage $\mathbf{m} \in \mathbb{N}^p$. The *polymatroidal ideal* $I_{\mathscr{P}} \subset R$ of $\mathscr{P}$ is the monomial ideal generated by the monomials corresponding to the lattice points in the base polytope $B(\mathscr{P})$ of $\mathscr{P}$. For each $i \geqslant 0$, the i-th *homological shift ideal* $\mathrm{HS}_i(I_{\mathscr{P}}) \subset R$ of $I_{\mathscr{P}}$ is the monomial ideal generated by the monomials corresponding to the shifts in the i-th position of the minimal free R-resolution of $I_{\mathscr{P}}$.

Let $I(\mathscr{P})$ be the independence polytope of $\mathscr{P}$. The *Möbius function* $\mu_{\mathscr{P}}\colon \mathbb{Z}^p \to \mathbb{Z}$ of the polymatroid $\mathscr{P}$ is defined inductively by setting $\mu_{\mathscr{P}}(\mathbf{n}) = 1$ if $\mathbf{n} \in B(\mathscr{P})$ and

$$\mu_{\mathscr{P}}(\mathbf{n}) \;=\; 1 - \sum_{\mathbf{w} \in \left(\mathbf{n} + \mathbb{Z}^p_{>0}\right) \cap I(\mathscr{P})} \mu_{\mathscr{P}}(\mathbf{w})$$

if $\mathbf{n} \in I(\mathscr{P}) \setminus B(\mathscr{P})$. For all $\mathbf{n} \in \mathbb{Z}^p \setminus I(\mathscr{P})$, we set $\mu_{\mathscr{P}}(\mathbf{n}) = 0$. The *Möbius support* of $\mathscr{P}$ is defined as $\mu\text{-supp}(\mathscr{P}) = \{\mathbf{n} \in \mathbb{N}^p \mid \mu_{\mathscr{P}}(\mathbf{n}) \neq 0\}$.

The main goal of this paper is to settle the following two conjectures regarding polymatroids.

Conjecture 1.1 (Bandari – Bayati – Herzog [Bay18, HMRZ21]). *All the homological shift ideals* $\mathrm{HS}_i(I_{\mathscr{P}})$ *of* $I_{\mathscr{P}}$ *are again polymatroidal ideals.*

Conjecture 1.2 (Castillo – Cid-Ruiz – Mohammadi – Montaño [CCRMM22]). *The Möbius support of* $\mathscr{P}$ *is a generalized polymatroid* (*i.e., a homogenization of it yields a polymatroid*).

Conjecture 1.1 has been verified in the following cases: in [Bay18], if $\mathscr{P}$ is a matroid; in [HMRZ21], if $\mathscr{P}$ satisfies the strong exchange property; in [FH23], if $\mathscr{P}$ has rank two; see also [Fic22]. In [CCRMM22, Theorem 7.17], the conclusion of Conjecture 1.2 was proven in the case where $\mathscr{P}$ is realizable, thus serving as motivation to state this conjecture. By [CCRMM22, Theorem 7.19] or [EL23, Remark 3.5], we know that Conjecture 1.2 holds when $\mathscr{P}$ is a matroid.

The K-ring of a matroid was recently introduced by Larson, Li, Payne, and Proudfoot [LLPP24]. Since the K-ring of a matroid has already become an object of interest, we are also interested in a K-theoretic description of the polymatroid $\mathscr{P}$. Let $\mathscr{M}$ be a matroid on a ground set E with subsets $\mathcal{S}_1,\ldots,\mathcal{S}_p \subseteq E$ such that the restriction polymatroid is $\mathscr{P}$. By considering the augmented K-ring of $\mathscr{M}$, we say that the *Snapper polynomial* of $\mathscr{P}$ is given by

$$\mathrm{Snapp}_{\mathscr{P}}(t_1,\ldots,t_p) = \chi\left(\mathscr{M}, \mathcal{L}_{\mathcal{S}_1}^{\otimes t_1} \otimes \cdots \otimes \mathcal{L}_{\mathcal{S}_p}^{\otimes t_p}\right).$$

For more details, see Definition 2.15, Definition 2.16, and Definition 2.17.

Motivated by the combinatorial notion of *caves* introduced in [CCRMM22], we introduce the *cave polynomial* of a polymatroid. The cave polynomial of $\mathscr{P}$ is given by

$$\mathrm{cave}_{\mathscr{P}}(t_1,\ldots,t_p) = \sum_{\mathbf{n}\in\mathbb{N}^p \text{ and } |\mathbf{n}|=\mathrm{rk}(\mathscr{P})} \mathbb{1}_{\mathscr{P}}(\mathbf{n}) \prod_{i=1}^{p-1}\left(1-\max_{i<j}\left\{\mathbb{1}_{\mathscr{P}}\left(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j\right)\right\}t_i^{-1}\right)\mathbf{t}^{\mathbf{n}},$$

where $\mathbb{1}_{\mathscr{P}}$ denotes the indicator function of the base polytope $B(\mathscr{P})$ of $\mathscr{P}$. It turns out that the Snapper polynomial $\mathrm{Snapp}_{\mathscr{P}}(t_1,\ldots,t_p)$ and the cave polynomial $\mathrm{cave}_{\mathscr{P}}(t_1,\ldots,t_p)$ encode the same information. Indeed, we have the equality

$$\mathrm{Snapp}_{\mathscr{P}}(t_1,\ldots,t_p) = \mathfrak{b}\left(\mathrm{cave}_{\mathscr{P}}(t_1,\ldots,t_p)\right),$$

where $\mathfrak{b}\colon \mathbb{Q}[t_1,\ldots,t_p] \to \mathbb{Q}[t_1,\ldots,t_p]$ is the $\mathbb{Q}$-linear map sending $t_1^{n_1}\cdots t_p^{n_p}$ to $\binom{t_1+n_1}{n_1}\cdots\binom{t_p+n_p}{n_p}$ (see (3)).

Our goal is to investigate various aspects of the cave polynomial. When $\mathscr{P}$ is realizable, our approach is to consider the corresponding multiplicity-free variety (see Remark 2.5). To address the general case (where $\mathscr{P}$ need not be realizable), our main idea is to show that the cave polynomial yields a *valuative function* on polymatroids. The theorem below contains our main results.

Theorem A. *Conjecture 1.1 and Conjecture 1.2 hold. More precisely, we have:*

(i) *The support of the cave polynomial* $\mathrm{cave}_{\mathscr{P}}(t_1,\ldots,t_p)$ *of* $\mathscr{P}$ *is a generalized polymatroid.*

(ii) *The cave polynomial* $\mathrm{cave}_{\mathscr{P}}(t_1,\ldots,t_p)$ *of* $\mathscr{P}$ *satisfies the equality*

$$\mathrm{cave}_{\mathscr{P}}(t_1,\ldots,t_p) = \sum_{\mathbf{n}\in\mathbb{N}^p} \mu_{\mathscr{P}}(\mathbf{n})\, t_1^{n_1}\cdots t_p^{n_p}.$$

In particular, Conjecture 1.2 holds.

(iii) *The* K*-polynomial of the polymatroidal ideal* $I_{\mathscr{P}} \subset R$ *is given by*

$$\mathcal{K}(I_{\mathscr{P}};t_1,\ldots,t_p) = t_1^{m_1}\cdots t_p^{m_p}\,\mathrm{cave}_{\mathscr{P}^\vee}\left(t_1^{-1},\ldots,t_p^{-1}\right),$$

where $\mathscr{P}^\vee = \mathbf{m} - \mathscr{P}$ is the dual polymatroid with respect to the cage $\mathbf{m}$*. Thus the* i*-th homological shift ideal of* $I_\mathscr{P}$ *is given by*

$$\mathrm{HS}_i(I_\mathscr{P}) \;=\; \left(x_1^{n_1}\cdots x_p^{n_p} \;\mid\; \mathbf{n}\in\mathbb{N}^p,\ |\mathbf{n}| = \mathrm{rk}(\mathscr{P})+i \ \text{ and } \ \mu_{\mathscr{P}^\vee}(\mathbf{m}-\mathbf{n}) \neq 0\right).$$

In particular, Conjecture 1.1 holds.

(iv) *The function* $\mathscr{P} \mapsto \mathrm{cave}_\mathscr{P}(t_1,\ldots,t_p)$ *assigning the cave polynomial to a polymatroid is valuative.*

2. Proofs of our results

Let $\mathscr{P}$ be a polymatroid on $[p] = \{1,\ldots,p\}$ with rank function $\mathrm{rk}_\mathscr{P}\colon 2^{[p]} \to \mathbb{Z}$. Let $\mathbf{m} = (m_1,\ldots,m_p) \in \mathbb{N}^p$ be a cage for the polymatroid $\mathscr{P}$. This means that

$$\mathrm{rk}_\mathscr{P}(\{i\}) \;\leqslant\; m_i \qquad \text{for all} \qquad 1\leqslant i\leqslant p.$$

Let $\Bbbk$ be a field and $R = \Bbbk[x_1,\ldots,x_p]$ be a standard $\mathbb{N}^p$-graded polynomial ring with $\deg(x_i) = \mathbf{e}_i \in \mathbb{N}^p$ for every i. Let $S = \Bbbk\left[x_{i,j} \mid 1\leqslant i\leqslant p, 0\leqslant j\leqslant m_i\right]$ be a standard $\mathbb{N}^p$-graded polynomial ring with $\deg(x_{i,j}) = \mathbf{e}_i \in \mathbb{N}^p$ for every i,j. We note that

$$\mathrm{MultiProj}(S) \;=\; \mathbb{P} \;:=\; \mathbb{P}_\Bbbk^{m_1}\times_\Bbbk\cdots\times_\Bbbk\mathbb{P}_\Bbbk^{m_p}$$

is the product of projective spaces associated to S.

The *base polytope* of the polymatroid $\mathscr{P}$ is given by

$$B(\mathscr{P}) \;:=\; \left\{\ \mathbf{v} = (v_1,\ldots,v_p)\in\mathbb{R}^p_{\geqslant 0} \;\middle|\; \textstyle\sum_{i=1}^p v_i = \mathrm{rk}(\mathscr{P}) \ \text{ and } \ \sum_{j\in J} v_j \leqslant \mathrm{rk}(J) \text{ for all } J\subseteq[p]\ \right\}.$$

The *independence polytope* of $\mathscr{P}$ is defined as

$$I(\mathscr{P}) \;:=\; \left\{\ \mathbf{v} = (v_1,\ldots,v_p)\in\mathbb{R}^p_{\geqslant 0} \;\middle|\; \textstyle\sum_{j\in J} v_j \leqslant \mathrm{rk}(J) \text{ for all } J\subseteq[p]\ \right\}.$$

We have the following equality

$$I(\mathscr{P}) \;=\; \left(B(\mathscr{P}) \,+\, \mathbb{R}^p_{\leqslant 0}\right) \,\cap\, \mathbb{R}^p_{\geqslant 0},$$

where $+$ denotes the Minkowski sum.

Our two objects of interest are the following.

Definition 2.1. (i) The *polymatroidal ideal* $I_\mathscr{P} \subset R$ of the polymatroid $\mathscr{P}$ is the monomial ideal given by

$$I_\mathscr{P} \;:=\; \left(\mathbf{x}^\mathbf{n} = x_1^{n_1}\cdots x_p^{n_p} \mid \mathbf{n}\in B(\mathscr{P})\cap\mathbb{N}^p\right).$$

(ii) The *Möbius function* $\mu_\mathscr{P}\colon \mathbb{Z}^p \to \mathbb{Z}$ of the polymatroid $\mathscr{P}$ is defined inductively by setting $\mu_\mathscr{P}(\mathbf{n}) := 1$ if $\mathbf{n}\in B(\mathscr{P})$ and

$$\mu_\mathscr{P}(\mathbf{n}) \;:=\; 1 - \sum_{\mathbf{w}\in\left(\mathbf{n}+\mathbb{Z}^p_{>0}\right)\cap I(\mathscr{P})} \mu_\mathscr{P}(\mathbf{w})$$

if $\mathbf{n}\in I(\mathscr{P})\setminus B(\mathscr{P})$. When $\mathbf{n}\notin I(\mathscr{P})$, we set $\mu_\mathscr{P}(\mathbf{n}) := 0$. Then the *Möbius support* of $\mathscr{P}$ is defined as

$$\mu\text{-}\mathrm{supp}(\mathscr{P}) \;:=\; \left\{\mathbf{n}\in\mathbb{N}^p \mid \mu_\mathscr{P}(\mathbf{n})\neq 0\right\}.$$

Consider the minimal $\mathbb{Z}^p$-graded free R-resolution

$$\mathbb{F}_\bullet: \quad \cdots \to F_i = \bigoplus_{j=1}^{\beta_i} R(-\mathbf{b}_{i,j}) \to \cdots \to F_0 \to I_{\mathscr{P}} \to 0$$

of $I_{\mathscr{P}}$, where each $\mathbf{b}_{i,j} = (b_{i,j,1}, \ldots, b_{i,j,p}) \in \mathbb{N}^p$. The i-th *homological shift ideal* of $I_{\mathscr{P}}$ is given by

$$\mathrm{HS}_i(I_{\mathscr{P}}) := \left(\mathbf{x}^{\mathbf{b}_{i,j}} \mid 1 \leqslant j \leqslant \beta_i\right) \subset R.$$

Notice that the equality $\mathrm{HS}_0(I_{\mathscr{P}}) = I_{\mathscr{P}}$ holds.

Definition 2.2. The K-*polynomial* of $I_{\mathscr{P}}$ is defined as

$$\mathcal{K}(I_{\mathscr{P}}; t_1, \ldots, t_p) := \sum_{i \geqslant 0} (-1)^i \sum_{j=1}^{\beta_i} \mathbf{t}^{\mathbf{b}_{i,j}} \in \mathbb{Z}[t_1, \ldots, t_p]$$

(see [MS05], [KM05]).

Remark 2.3. By an abuse of notation, we also denote by $\mathscr{P}$ the associated base discrete polymatroid (i.e., the lattice points in $B(\mathscr{P}) \cap \mathbb{N}^p$). Being a base discrete polymatroid is equivalent to being an *M-convex set* in the sense of Murota [Mur03].

We shall need the following "dual version" of the aforementioned polymatroidal ideal.

Definition 2.4. The *dual polymatroidal ideal* $J_{\mathscr{P}} \subset S$ of $\mathscr{P}$ with respect to the cage $\mathbf{m}$ is given by

$$J_{\mathscr{P}} := \bigcap_{\mathbf{n} \in B(\mathscr{P}) \cap \mathbb{N}^p} \mathfrak{p}_{\mathbf{m}-\mathbf{n}} = \bigcap_{\mathbf{n} \in B(\mathscr{P}) \cap \mathbb{N}^p} \left(x_{i,j} \mid 1 \leqslant i \leqslant p \text{ and } 0 \leqslant j < m_i - n_i\right).$$

The *polymatroidal multiprojective variety* of $\mathscr{P}$ with respect to the cage $\mathbf{m} = (m_1, \ldots, m_p)$ is given by

$$Y_{\mathscr{P}} := V(J_{\mathscr{P}}) \subset \mathbb{P} = \mathbb{P}_{\Bbbk}^{m_1} \times_{\Bbbk} \cdots \times_{\Bbbk} \mathbb{P}_{\Bbbk}^{m_p}.$$

Remark 2.5 ($\Bbbk$ infinite)**.** Our motivation to consider the multiprojective variety $Y_{\mathscr{P}} \subset \mathbb{P}$ comes from the following algebro-geometric ideas that are available in the realizable case. If $\mathscr{P}$ is realizable (i.e., linear over $\Bbbk$), then we can find a *multiplicity-free* subvariety $X_{\mathscr{P}} \subset \mathbb{P}$ such that the support of its multidegrees is given by $\mathscr{P}$ (see [CCRMM22, Proposition 7.15]). Then a remarkable result of Brion [Bri03] yields a flat degeneration of $X_{\mathscr{P}}$ to $Y_{\mathscr{P}}$. This means that the multigraded generic initial ideal of the prime associated to $X_{\mathscr{P}}$ is square-free and coincides with $J_{\mathscr{P}}$ (see [CCRC23, Theorem D]).

Remark 2.6. We say that the support of a polynomial $f(t_1, \ldots, t_p) \in \mathbb{R}[t_1, \ldots, t_p]$ is a *generalized polymatroid* if the support of the homogeneous polynomial $t_0^{\deg(f)} f(\frac{t_1}{t_0}, \ldots, \frac{t_p}{t_0}) \in \mathbb{R}[t_0, t_1, \ldots, t_p]$ is a (base discrete) polymatroid.

Remark 2.7. When $\mathscr{P}$ is a matroid, $J_{\mathscr{P}}$ is the *"matroid ideal"* studied in [NPS02].

Remark 2.8. The set $\mathscr{P}^\vee := \mathbf{m} - \mathscr{P} = \{\mathbf{m} - \mathbf{n} \mid \mathbf{n} \in \mathscr{P}\}$ is also a polymatroid. We call it the *dual polymatroid* of $\mathscr{P}$ with respect to the cage $\mathbf{m}$. The rank function of the dual polymatroid $\mathscr{P}^\vee$ is given by

$$\mathrm{rk}_{\mathscr{P}^\vee}(J) := \sum_{j \in J} m_j + \mathrm{rk}_{\mathscr{P}}([p] \setminus J) - \mathrm{rk}_{\mathscr{P}}([p]) \quad \text{for all } J \subseteq [p]$$

(see [Sch03, §44.6f]). Moreover, we have $\mathscr{P}^{\vee\vee} = \mathscr{P}$.

Remark 2.9. The Chow ring of $\mathbb{P}$ and the Grothendieck ring of coherent sheaves on $\mathbb{P}$ are given by

$$A^*(\mathbb{P}) \;\cong\; \frac{\mathbb{Z}[t_1,\ldots,t_p]}{\left(t_1^{m_1+1},\ldots,t_p^{m_p+1}\right)} \quad\text{and}\quad K(\mathbb{P}) \;\cong\; \frac{\mathbb{Z}[t_1,\ldots,t_p]}{\left((1-t_1)^{m_1+1},\ldots,(1-t_p)^{m_p+1}\right)}.$$

For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}$, we can write

$$[\mathcal{F}] \;=\; \sum_{\mathbf{n}\in\mathbb{N}^p \text{ and } |\mathbf{n}|\leqslant\dim(\mathrm{Supp}(\mathcal{F}))} c_{\mathbf{n}}(\mathcal{F})\left[\mathcal{O}_{\mathbb{P}_{\Bbbk}^{n_1}\times_{\Bbbk}\cdots\times_{\Bbbk}\mathbb{P}_{\Bbbk}^{n_p}}\right] \;\in\; K(\mathbb{P}).$$

For any closed subscheme $X\subset\mathbb{P}$, we set $c_{\mathbf{n}}(X) := c_{\mathbf{n}}(\mathcal{O}_X)$. Since by construction $\dim(Y_{\mathscr{P}}) = \mathrm{rk}(\mathscr{P})$, we can write the class $[\mathcal{O}_{Y_{\mathscr{P}}}]\in K(\mathbb{P})$ as

$$[\mathcal{O}_{Y_{\mathscr{P}}}] \;=\; \sum_{\mathbf{n}\in\mathbb{N}^p \text{ and } |\mathbf{n}|\leqslant\mathrm{rk}(\mathscr{P})} c_{\mathbf{n}}(Y_{\mathscr{P}})\left[\mathcal{O}_{\mathbb{P}_{\Bbbk}^{n_1}\times_{\Bbbk}\cdots\times_{\Bbbk}\mathbb{P}_{\Bbbk}^{n_p}}\right] \;\in\; K(\mathbb{P}).$$

Under the above isomorphism describing $K(\mathbb{P})$, we can also write

$$[\mathcal{O}_{Y_{\mathscr{P}}}] \;=\; \sum_{\mathbf{n}\in\mathbb{N}^p \text{ and } |\mathbf{n}|\leqslant\mathrm{rk}(\mathscr{P})} c_{\mathbf{n}}(Y_{\mathscr{P}})\,(1-t_1)^{m_1-n_1}\cdots(1-t_p)^{m_p-n_p} \;\in\; K(\mathbb{P}).$$

Then we obtain

$$[Y_{\mathscr{P}}] \;=\; \sum_{\mathbf{n}\in\mathbb{N}^p \text{ and } |\mathbf{n}|=\mathrm{rk}(\mathscr{P})} c_{\mathbf{n}}(Y_{\mathscr{P}})\, t_1^{m_1-n_1}\cdots t_p^{m_p-n_p} \;\in\; A^*(\mathbb{P})$$

(i.e., when $|\mathbf{n}| = \dim(Y_{\mathscr{P}})$, the constants $c_{\mathbf{n}}(Y_{\mathscr{P}}) = \deg_{\mathbb{P}}^{\mathbf{n}}(Y_{\mathscr{P}})$ encode the multidegrees of $Y_{\mathscr{P}}$).

The next technical proposition relates the previous invariants we have seen.

Proposition 2.10. *Under the above notation, the following statements hold:*

(i) $\mu_{\mathscr{P}}(\mathbf{n}) = c_{\mathbf{n}}(Y_{\mathscr{P}})$ *for all* $\mathbf{n}\in\mathbb{N}^p$.

(ii) *In terms of the dual polymatroid* $\mathscr{P}^\vee = \mathbf{m}-\mathscr{P}$, *we have the equality*

$$\mathcal{K}(I_{\mathscr{P}^\vee};\mathbf{t}) \;=\; \sum_{\mathbf{n}\in\mathbb{N}^p} c_{\mathbf{n}}(Y_{\mathscr{P}})\, t_1^{m_1-n_1}\cdots t_p^{m_p-n_p}.$$

Proof. (i) This part follows from [Knu09] (see also [CCRMM22]).

(ii) Consider the K-polynomial $\mathcal{K}(S/J_{\mathscr{P}};\mathbf{t})$ of $S/J_{\mathscr{P}}$. Since each minimal prime of $J_{\mathscr{P}}$ is of the form $\mathfrak{p}_{\mathbf{m}-\mathbf{n}}$ (a Borel-fixed prime in a multigraded setting), one can show that the K-polynomial $\mathcal{K}(S/J_{\mathscr{P}};\mathbf{t})\in\mathbb{Z}[t_1,\ldots,t_p]$ and the class $[\mathcal{O}_{Y_{\mathscr{P}}}]\in K(\mathbb{P})$ determine one another; that is, we have the equality

$$\mathcal{K}(S/J_{\mathscr{P}};\mathbf{t}) \;=\; \sum_{\mathbf{n}\in\mathbb{N}^p} c_{\mathbf{n}}(Y_{\mathscr{P}})\,(1-t_1)^{m_1-n_1}\cdots(1-t_p)^{m_p-n_p} \;\in\; \mathbb{Z}[t_1,\ldots,t_p]$$

(see [CCRMM22, §4]). The Alexander dual of $J_{\mathscr{P}}\subset S$ is the monomial ideal $K_{\mathscr{P}}\subset S$ given by

$$K_{\mathscr{P}} \;:=\; \left(\, \mathbf{x}_{\mathbf{m}-\mathbf{n}} = \textstyle\prod_{1\leqslant i\leqslant p,\,0\leqslant j<m_i-n_i} x_{i,j} \;\middle|\; \mathbf{n}\in B(\mathscr{P})\cap\mathbb{N}^p \,\right)$$

(see [HH11, Corollary 1.5.5]). By [MS05, Theorem 5.14], we have the equality

$$\mathcal{K}(K_{\mathscr{P}};\mathbf{t}) \;=\; \mathcal{K}(S/J_{\mathscr{P}};\mathbf{1}-\mathbf{t}) \;=\; \sum_{\mathbf{n}\in\mathbb{N}^p} c_{\mathbf{n}}(Y_{\mathscr{P}})\, t_1^{m_1-n_1}\cdots t_p^{m_p-n_p} \;\in\; \mathbb{Z}[t_1,\ldots,t_p].$$

Notice that $K_{\mathscr{P}}$ can be seen naturally as the polarization of $I_{\mathscr{P}^\vee}$ by mapping the monomial $\mathbf{x}^{\mathbf{m}-\mathbf{n}} = x_1^{m_1-n_1}\cdots x_p^{m_p-n_p}$ in R to the monomial $\mathbf{x}_{\mathbf{m}-\mathbf{n}} = \prod_{1\leqslant i\leqslant p,\,0\leqslant j<m_i-n_i} x_{i,j}$ in S. Finally, by standard

properties of polarization (see [HH11, §1.6]), it follows that $\mathcal{K}(I_{\mathscr{P}^\vee};\mathbf{t}) = \mathcal{K}(K_{\mathscr{P}};\mathbf{t})$. This concludes the proof of the proposition. □

We now recall the notion of *valuative functions* on polymatroids.

Definition 2.11. The *indicator function* $\mathbb{1}_{\mathscr{P}}\colon \mathbb{R}^p \to \mathbb{Z}$ of a polymatroid $\mathscr{P}$ is the function given by

$$\mathbb{1}_{\mathscr{P}}(\mathbf{v}) := \begin{cases} 1 & \text{if } \mathbf{v} \in B(\mathscr{P}) \\ 0 & \text{otherwise.} \end{cases}$$

The *valuative group* of polymatroids on $[p]$ with cage $\mathbf{m} = (m_1, \ldots, m_p)$, denoted $\mathrm{Val}_{\mathbf{m}}$, is the subgroup of $\mathrm{Hom}_{\mathrm{Sets}}(\mathbb{R}^p, \mathbb{Z})$ generated by all the indicator functions $\mathbb{1}_{\mathscr{P}}$ for $\mathscr{P}$ a polymatroid on $[p]$ with cage $\mathbf{m}$. A function $f\colon \mathbb{Pol}_{\mathbf{m}} \to G$ from the set $\mathbb{Pol}_{\mathbf{m}}$ of polymatroids with cage $\mathbf{m}$ to an Abelian group G is said to be *valuative* if it factors through $\mathrm{Val}_{\mathbf{m}}$. This means that, for all $\mathscr{P}_1, \ldots, \mathscr{P}_k \in \mathbb{Pol}_{\mathbf{m}}$ and all $a_1, \ldots, a_k \in \mathbb{Z}$, if $\sum_{i=1}^k a_i \mathbb{1}_{\mathscr{P}_i} = 0 \in \mathrm{Hom}_{\mathrm{Sets}}(\mathbb{R}^p, \mathbb{Z})$, then $\sum_{i=1}^k a_i f(\mathscr{P}_i) = 0 \in G$.

Remark 2.12. From [DF10] or [EL24, Remark 3.16], the valuative group $\mathrm{Val}_{\mathbf{m}}$ is generated by the indicator functions of realizable polymatroids over $\mathbb{C}$. Therefore if two valuative functions $f, g\colon \mathbb{Pol}_{\mathbf{m}} \to G$ agree on realizable polymatroids, then they are equal.

Our approach is based on defining the following polynomial and showing that it is *valuative*. We call this polynomial the *cave polynomial* because it is motivated by the combinatorial notion of *caves* introduced in [CCRMM22].

Definition 2.13. The *cave polynomial* of the polymatroid $\mathscr{P}$ is given by

$$\mathrm{cave}_{\mathscr{P}}(t_1, \ldots, t_p) := \sum_{\mathbf{n} \in \mathbb{N}^p \text{ and } |\mathbf{n}| = \mathrm{rk}(\mathscr{P})} \mathbb{1}_{\mathscr{P}}(\mathbf{n}) \prod_{i=1}^{p-1} \left(1 - \max_{i<j} \{\mathbb{1}_{\mathscr{P}}(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)\} t_i^{-1}\right) \mathbf{t}^{\mathbf{n}}.$$

Notice that $\mathrm{cave}_{\mathscr{P}}(t_1, \ldots, t_p)$ is an honest polynomial in $\mathbb{Z}[t_1, \ldots, t_p]$ and not a Laurent polynomial with possibly negative exponents of the variables t_i.

Remark 2.14. Write $\mathrm{cave}_{\mathscr{P}}(\mathbf{t}) = \sum_{|\mathbf{n}| \leqslant \mathrm{rk}(\mathscr{P})} a_{\mathbf{n}}(\mathscr{P}) \mathbf{t}^{\mathbf{n}}$. By ordering the points in $B(\mathscr{P}) \cap \mathbb{N}^p$ with respect to the lexicographic order (with $1 < 2 < \cdots < p$), we obtain a shelling of the facets of the simplicial complex $\Delta(J_{\mathscr{P}})$ associated to $J_{\mathscr{P}}$ (see [CCRMM22, proof of Lemma 6.8]). Then by [CCRMM22, Proposition 4.6], we obtain that the coefficients of the cave polynomial $\mathrm{cave}_{\mathscr{P}}(\mathbf{t})$ describe the class $[\mathcal{O}_{Y_{\mathscr{P}}}] \in K(\mathbb{P})$; that is,

$$[\mathcal{O}_{Y_{\mathscr{P}}}] = \sum_{\mathbf{n} \in \mathbb{N}^p \text{ and } |\mathbf{n}| \leqslant \mathrm{rk}(\mathscr{P})} a_{\mathbf{n}}(\mathscr{P}) \left[\mathcal{O}_{\mathbb{P}_{\Bbbk}^{n_1} \times_{\Bbbk} \cdots \times_{\Bbbk} \mathbb{P}_{\Bbbk}^{n_p}}\right].$$

Hence we have the equalities

$$a_{\mathbf{n}}(\mathscr{P}) = c_{\mathbf{n}}(Y_{\mathscr{P}}) = \mu_{\mathscr{P}}(\mathbf{n})$$

(see Remark 2.9 and Proposition 2.10). As a consequence, we can write

$$\mathrm{cave}_{\mathscr{P}}(t_1, \ldots, t_p) = \sum_{\mathbf{n} \in \mathbb{N}^p} \mu_{\mathscr{P}}(\mathbf{n}) t_1^{n_1} \cdots t_p^{n_p}.$$

By symmetry, since we can choose any lexicographic order on $[p]$, we get

$$\mathrm{cave}_{\mathscr{P}}(t_1,\dots,t_p) \;:=\; \sum_{\mathbf{n}\in\mathbb{N}^p \text{ and } |\mathbf{n}|=\mathrm{rk}(\mathscr{P})} \mathbb{1}_{\mathscr{P}}(\mathbf{n}) \prod_{i=1}^{p-1}\left(1-\max_{i<j}\left\{\mathbb{1}_{\mathscr{P}}\left(\mathbf{n}-\mathbf{e}_{\pi(i)}+\mathbf{e}_{\pi(j)}\right)\right\}t_{\pi(i)}^{-1}\right)\mathbf{t}^{\mathbf{n}}$$

for any permutation $\pi \in \mathfrak{S}_p$ on $[p]$. Let $\mathfrak{b}\colon \mathbb{Q}[t_1,\dots,t_p] \to \mathbb{Q}[t_1,\dots,t_p]$ be the $\mathbb{Q}$-linear map sending $t_1^{n_1}\cdots t_p^{n_p}$ to $\binom{t_1+n_1}{n_1}\cdots\binom{t_p+n_p}{n_p}$. The cave polynomial $\mathrm{cave}_{\mathscr{P}}(t_1,\dots,t_p)$ satisfies the equation

$$\chi\big(Y_{\mathscr{P}}, \mathcal{O}_{Y_{\mathscr{P}}}(\nu_1,\dots,\nu_p)\big) \;=\; \left(\mathfrak{b}\left(\mathrm{cave}_{\mathscr{P}}\right)\right)(\nu_1,\dots,\nu_p) \tag{1}$$

for all $(\nu_1,\dots,\nu_p)\in\mathbb{Z}^p$.

We are also interested in the K-ring of a matroid and in the notion of multisymmetric lift.

Definition 2.15 ([LLPP24]; see also [EL23, §2.2]). Let $\mathscr{M}$ be a matroid on the ground set E. Let $K(\mathscr{M})$ be the *augmented* K*-ring* of $\mathscr{M}$, as introduced in [LLPP24]. We are interested in the following features of $K(\mathscr{M})$:

- (i) It is endowed with an *Euler characteristic map* $\chi(\mathscr{M},-)\colon K(\mathscr{M}) \to \mathbb{Z}$.
- (ii) Each nonempty subset $\mathcal{S} \subseteq E$ defines an element $[\mathcal{L}_{\mathcal{S}}] \in K(\mathscr{M})$.
- (iii) The elements $\{[\mathcal{L}_{\mathcal{S}}]\}_{\emptyset\subsetneq\mathcal{S}\subseteq E}$ generate $K(\mathscr{M})$ as a ring.
- (iv) A *line bundle* in $K(\mathscr{M})$ is a Laurent monomial in the $[\mathcal{L}_{\mathcal{S}}]$.

Definition 2.16 ([EL24, EL23, CHL⁺22]). The *multisymmetric lift* of $\mathscr{P}$ is a matroid $\mathscr{M}$ on a ground set E which is equipped with a distinguished partition $E = \mathcal{S}_1 \sqcup\cdots\sqcup \mathcal{S}_p$ satisfying the following properties:

- (i) $|\mathcal{S}_i| = m_i$ for each $1 \leqslant i \leqslant p$.
- (ii) $\mathrm{rk}_{\mathscr{M}}\colon 2^E \to \mathbb{N}$ is preserved by the action of the product of symmetric groups $\mathfrak{S}_{\mathcal{S}_1}\times\cdots\times\mathfrak{S}_{\mathcal{S}_p}$.
- (iii) For each $J \subseteq [p]$, we have

$$\mathrm{rk}_{\mathscr{P}}(J) \;=\; \mathrm{rk}_{\mathscr{M}}\left(\bigsqcup_{j\in J}\mathcal{S}_j\right).$$

The multisymmetric lift $\mathscr{M}$ always exists (see [CHL⁺22, Theorem 2.11]). We say that $\mathscr{M}$ is a matroid on a ground set E with subsets $\mathcal{S}_1,\dots,\mathcal{S}_p \subseteq E$ such that the *restriction polymatroid* is $\mathscr{P}$.

Let $\mathscr{M}$ be a matroid on a ground set E with subsets $\mathcal{S}_1,\dots,\mathcal{S}_p \subseteq E$ such that the restriction polymatroid is $\mathscr{P}$. By [EL23, Theorem 1.2], the *Snapper polynomial* of $\mathcal{L}_{\mathcal{S}_1},\dots,\mathcal{L}_{\mathcal{S}_p}$ satisfies the following equality

$$\chi\left(\mathscr{M},\mathcal{L}_{\mathcal{S}_1}^{\otimes\nu_1}\otimes\cdots\otimes\mathcal{L}_{\mathcal{S}_p}^{\otimes\nu_p}\right) \;=\; \chi\left(Y_{\mathscr{P}}, \mathcal{O}_{Y_{\mathscr{P}}}(\mathbf{v})\right) \tag{2}$$

for all $\mathbf{v} = (\nu_1,\dots,\nu_p) \in \mathbb{Z}^p$. Since the right-hand side of (2) depends only on $\mathscr{P}$, we can make the following definition.

Definition 2.17. The *Snapper polynomial* of the polymatroid $\mathscr{P}$ is given by

$$\mathrm{Snapp}_{\mathscr{P}}(t_1,\dots,t_p) \;:=\; \chi\left(\mathscr{M},\mathcal{L}_{\mathcal{S}_1}^{\otimes t_1}\otimes\cdots\otimes\mathcal{L}_{\mathcal{S}_p}^{\otimes t_p}\right) \;\in\; \mathbb{N}[t_1,\dots,t_p].$$

We have the following explicit relation between the Snapper polynomial and the cave polynomial

$$\mathrm{Snapp}_{\mathscr{P}}(t_1,\dots,t_p) \;=\; \mathfrak{b}\big(\mathrm{cave}_{\mathscr{P}}(t_1,\dots,t_p)\big). \tag{3}$$

Indeed, the equality follows from (1) and (2).

The next proposition is invaluable for our approach.

Proposition 2.18. *The function* $\mathrm{cave}\colon \mathbb{P}\mathrm{ol}_{\mathbf{m}} \to \mathbb{Z}[t_1,\ldots,t_p]$, $\mathscr{P} \mapsto \mathrm{cave}_{\mathscr{P}}(t_1,\ldots,t_p)$ *assigning the cave polynomial to a polymatroid is valuative.*

Proof. Due to Remark 2.8, Remark 2.14, and Proposition 2.10, it suffices to show the valuativity of the function assigning to each polymatroid $\mathscr{P}$ the $\mathbb{N}^p$-graded Hilbert function of the polymatroidal ideal $I_{\mathscr{P}} \subset R$. For all $\mathbf{n} \in \mathbb{N}^p$, we have that $\dim_{\mathbb{k}}([I_{\mathscr{P}}]_{\mathbf{n}}) \neq 0$ if and only if $\mathbf{n}$ belongs to the region

$$\bigcup_{\mathbf{w}\in B(\mathscr{P})\cap\mathbb{N}^p} \left(\mathbf{w} + \mathbb{Z}^p_{\geqslant 0}\right).$$

Equivalently, we obtain

$$\dim_{\mathbb{k}}([I_{\mathscr{P}}]_{\mathbf{n}}) \;=\; i_{\mathbf{n}+\mathbb{R}^p_{\leqslant 0}}(\mathscr{P}) \;:=\; \begin{cases} 1 & \text{if } B(\mathscr{P}) \cap \left(\mathbf{n}+\mathbb{R}^p_{\leqslant 0}\right) \neq \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

Finally, from [AFR10, Corollary 4.3], we know that the function $i_{\mathbf{n}+\mathbb{R}^p_{\leqslant 0}}\colon \mathbb{P}\mathrm{ol}_{\mathbf{m}} \to \mathbb{Z}$ is valuative. (The statement of [AFR10, Corollary 4.3] is for matroids, but the same proof holds for polymatroids.) □

Lemma 2.19. *For any* $\mathbf{b} \in \mathbb{N}^p$, *the set* $\mathscr{P}' = \mathscr{P} - \mathbf{b} = \{\mathbf{n} - \mathbf{b} \mid \mathbf{n} \in \mathscr{P} \text{ and } \mathbf{n} \geqslant \mathbf{b}\}$ *and the truncation* $\mathscr{P}_{\mathbf{b}} = \{\mathbf{n} \in \mathscr{P} \mid \mathbf{n} \geqslant \mathbf{b}\}$ *are both* (*base discrete*) *polymatroids.*

Proof. Write $F_{\mathscr{P}}(\mathbf{t}) = \sum_{\mathbf{n}\in\mathscr{P}} \frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}$ for the generating function of $\mathscr{P} \subset \mathbb{N}^p$. By [BH20, Theorem 3.10], the polynomial $F_{\mathscr{P}}$ is Lorentzian. Now, by [RSW23, Proposition 3.3], the generating functions $F_{\mathscr{P}'}$ and $F_{\mathscr{P}_{\mathbf{b}}}$ are also Lorentzian. Another application of [BH20, Theorem 3.10] yields that $\mathscr{P}'$ and $\mathscr{P}_{\mathbf{b}}$ are M-convex sets. Hence, they are both (base discrete) polymatroids. □

Lemma 2.20. *Let* $i \in [p]$ *and consider* $\mathscr{P}' = \mathscr{P} - \mathbf{e}_i$ *and* $\mathscr{P}_{\mathbf{e}_i}$. *Then, for all* $\mathbf{n} \geqslant \mathbf{e}_i$, *we have the equalities*

$$c_{\mathbf{n}}(Y_{\mathscr{P}_{\mathbf{e}_i}}) \;=\; c_{\mathbf{n}-\mathbf{e}_i}(Y_{\mathscr{P}'}) \;=\; c_{\mathbf{n}}(Y_{\mathscr{P}}).$$

Proof. The equalities $c_{\mathbf{n}}(Y_{\mathscr{P}_{\mathbf{e}_i}}) = a_{\mathbf{n}}(\mathscr{P}_{\mathbf{e}_i}) = a_{\mathbf{n}-\mathbf{e}_i}(\mathscr{P}') = c_{\mathbf{n}-\mathbf{e}_i}(Y_{\mathscr{P}'})$ follow from Remark 2.14. We prove the other equality. Consider the functions $f, g\colon \mathbb{P}\mathrm{ol}_{\mathbf{m}} \to \mathbb{Z}$ given by $f(\mathscr{P}) := c_{\mathbf{n}}(Y_{\mathscr{P}})$ and $g(\mathscr{P}) := c_{\mathbf{n}-\mathbf{e}_i}(Y_{\mathscr{P}-\mathbf{e}_i})$. By Proposition 2.18, both functions are valuative. Thus, due to Remark 2.12, it suffices to show that f and g agree on realizable polymatroids.

Let $\mathscr{P}$ be a realizable polymatroid over $\mathbb{C}$. Due to [CCRMM22, Proposition 7.15] and Remark 2.5, we can find a multiplicity-free $X_{\mathscr{P}} \subset \mathbb{P}_{\mathbb{C}} = \mathbb{P}^{m_1}_{\mathbb{C}} \times \cdots \times \mathbb{P}^{m_p}_{\mathbb{C}}$ such that $f(\mathscr{P}) = c_{\mathbf{n}}(Y_{\mathscr{P}}) = c_{\mathbf{n}}(X_{\mathscr{P}})$. Let $H \subset \mathbb{P}_{\mathbb{C}}$ be the pullback of a general hyperplane in $\mathbb{P}^{m_i}_{\mathbb{C}}$. Then, by Bertini's theorem, we have that $X_{\mathscr{P}} \cap H$ is also a multiplicity-free variety and that $c_{\mathbf{n}-\mathbf{e}_i}(X_{\mathscr{P}} \cap H) = c_{\mathbf{n}}(X_{\mathscr{P}})$. Again, applying [CCRMM22, Proposition 7.15] to the polymatroid $\mathscr{P} - \mathbf{e}_i$, we obtain $g(\mathscr{P}) = c_{\mathbf{n}-\mathbf{e}_i}(Y_{\mathscr{P}-\mathbf{e}_i}) = c_{\mathbf{n}-\mathbf{e}_i}(X_{\mathscr{P}} \cap H)$. So the proof is complete. □

We are now ready to prove our main results.

Proof of Theorem A. (i) Set $\mathscr{C} := \mu\text{-supp}(\mathscr{P}) = \{\mathbf{n} \in \mathbb{N}^p \mid c_{\mathbf{n}}(\mathscr{P}) \neq 0\} = \mathrm{supp}(\mathrm{cave}_{\mathscr{P}}(t_1,\ldots,t_p))$ (see Remark 2.14 and Proposition 2.10). We show that $\mathscr{C}$ is a cave (see [CCRMM22, §5]) and so it is a generalized polymatroid by [CCRMM22, Theorem 5.18]. Let $\mathbf{b} \in \mathbb{N}^p$ and consider $\mathscr{A} := \mathscr{C}_{\mathbf{b}}$, the $\mathbf{b}$-truncation of $\mathscr{C}$. By Lemma 2.19, we have that $\mathscr{P}_{\mathbf{b}}$ is also a polymatroid. Iteratively applying Lemma 2.20, we get $c_{\mathbf{n}}(\mathscr{P}) = c_{\mathbf{n}}(\mathscr{P}_{\mathbf{b}}) = c_{\mathbf{n}-\mathbf{b}}(\mathscr{P}-\mathbf{b})$ for all $\mathbf{n} \geqslant \mathbf{b}$. Thus $\mathscr{A} = \operatorname{supp}(\operatorname{cave}_{\mathscr{P}-\mathbf{b}}(t_1,\ldots,t_p)) + \mathbf{b}$. We now check the conditions of [CCRMM22, Definition 5.8]:

- Part (a) holds because we already know that $\mathscr{A}^{\mathrm{top}} = \mathscr{P}_{\mathbf{b}}$ is a polymatroid.
- Part (b) holds by construction since the cave polynomial mimics the notion of stalactites.
- Part (c) holds by induction on the rank of $\mathscr{P}$ because the rank of $\mathscr{P}-\mathbf{b}$ is strictly smaller than the rank of $\mathscr{P}$ when $\mathbf{b} \neq \mathbf{0}$. The base case is clear since $\operatorname{cave}_{\mathscr{P}}(t_1,\ldots,t_p) = 1$ when $\mathscr{P} = \{\mathbf{0}\}$ is the polymatroid of rank zero.

Therefore, the support of the cave polynomial $\operatorname{cave}_{\mathscr{P}}(t_1,\ldots,t_p)$ is a cave, and so we are done with the proof of this part.

(ii) This part follows from Remark 2.14 and part (i).

(iii) The equality

$$\mathcal{K}(I_{\mathscr{P}}; t_1,\ldots,t_p) = t_1^{m_1}\cdots t_p^{m_p}\operatorname{cave}_{\mathscr{P}^\vee}\left(t_1^{-1},\ldots,t_p^{-1}\right)$$

follows from Proposition 2.10 and Remark 2.14. By part (i), we already know that the support of $\operatorname{cave}_{\mathscr{P}^\vee}(\mathbf{t})$ is a generalized polymatroid. This implies that the support of $\mathcal{K}(I_{\mathscr{P}};\mathbf{t}) = \mathbf{t}^{\mathbf{m}}\operatorname{cave}_{\mathscr{P}^\vee}(\mathbf{t}^{-1})$ is also a generalized polymatroid. Recall that polymatroidal ideals have a linear resolution (see [HH11, Theorem 12.6.2]). Hence $\mathrm{HS}_i(I_{\mathscr{P}})$ is generated by the monomials $\mathbf{x}^{\mathbf{n}} = x_1^{n_1}\cdots x_p^{n_p}$ of total degree $\operatorname{rk}(\mathscr{P})+i$ such that $\mathbf{t}^{\mathbf{n}} = t_1^{n_1}\cdots t_p^{n_p}$ belongs to the support of $\mathcal{K}(I_{\mathscr{P}};\mathbf{t})$. This implies the equality

$$\mathrm{HS}_i(I_{\mathscr{P}}) = \left(\mathbf{x}^{\mathbf{n}} \mid \mathbf{n} \in \mathbb{N}^p,\ |\mathbf{n}| = \operatorname{rk}(\mathscr{P})+i \text{ and } \mu_{\mathscr{P}^\vee}(\mathbf{m}-\mathbf{n}) \neq 0\right)$$

and shows that Conjecture 1.1 holds.

(iv) This part was proved in Proposition 2.18. □

We finish the paper with the following example.

Example 2.21. We illustrate Theorem A in an explicit example. To this end, consider the polymatroid $\mathscr{P}$ described in [PP23, Section 7]. It is a polymatroid on the set $[3] = \{1,2,3\}$ with cage $(2,2,4)$ and rank function $\operatorname{rk}_{\mathscr{P}}\colon 2^{[3]} \to \mathbb{N}$ given by

$$\operatorname{rk}(\varnothing) = 0, \quad \operatorname{rk}(\{1\}) = \operatorname{rk}(\{2\}) = 2, \quad \operatorname{rk}(\{3\}) = \operatorname{rk}(\{1,2\}) = 4,$$
$$\operatorname{rk}(\{1,3\}) = \operatorname{rk}(\{2,3\}) = \operatorname{rk}(\{1,2,3\}) = 5.$$

The base polytope $B(\mathscr{P})$ and the independence polytope $I(\mathscr{P})$ are shown in Figure 1. The lattice points in the base polytope are given by

$$B(\mathscr{P}) \cap \mathbb{N}^3 = \left\{(0,2,3), (2,0,3), (1,2,2), (2,1,2), (2,2,1), (1,1,3), (1,0,4), (0,1,4)\right\},$$

and thus the polymatroidal ideal $I_{\mathscr{P}} \subset \Bbbk[x_1,x_2,x_3]$ is given by

$$I_{\mathscr{P}} = \left(x_2^2x_3^3,\, x_1^2x_3^3,\, x_1x_2^2x_3^2,\, x_1^2x_2x_3^2,\, x_1^2x_2^2x_3,\, x_1x_2x_3^3,\, x_1x_3^4,\, x_2x_3^4\right).$$

The K-polynomial of $I_{\mathscr{P}}$ is given by

$$\begin{aligned}\mathcal{K}(I_{\mathscr{P}}; t_1,t_2,t_3) = {}& t_1^2t_2^2t_3^3 + t_1^2t_2t_3^4 + t_1t_2^2t_3^4 \\ & - 2t_1^2t_2^2t_3^2 - 2t_1^2t_2t_3^3 - 2t_1t_2^2t_3^3 - t_1^2t_3^4 - 2t_1t_2t_3^4 - t_2^2t_3^4\end{aligned}$$

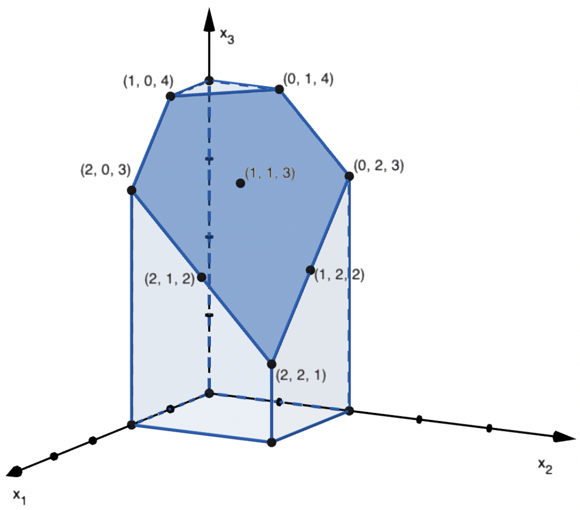

FIGURE 1. Base and independence polytopes of $\mathscr{P}$.

$$+t_1^2t_2^2t_3+t_1^2t_2t_3^2+t_1t_2^2t_3^2+t_1^2t_3^3+t_1t_2t_3^3+t_2^2t_3^3+t_1t_3^4+t_2t_3^4.$$

The dual polymatroid $\mathscr{P}^\vee$ of $\mathscr{P}$, with respect to the cage $(2,2,4)$, is described by the lattice points

$$B(\mathscr{P}^\vee)\cap\mathbb{N}^3 = \{(2,0,1), (0,2,1), (1,0,2), (0,1,2), (0,0,3), (1,1,1), (1,2,0), (2,1,0)\}.$$

The cave polynomial of $\mathscr{P}^\vee$ is given by

$$\begin{aligned}\mathrm{cave}_{\mathscr{P}^\vee}(t_1,t_2,t_3) &= t_1^2t_2+t_1t_2^2+t_1^2t_3+t_1t_2t_3+t_2^2t_3+t_1t_3^2+t_2t_3^2+t_3^3\\ &\quad -t_1^2-2t_1t_2-t_2^2-2t_1t_3-2t_2t_3-2t_3^2\\ &\quad +t_1+t_2+t_3.\end{aligned}$$

Using the SageMath [Sag25] function `is_lorentzian()`, we verified that the homogenization of the (sign-changed) polynomials $\mathcal{K}(I_{\mathscr{P}};t_1,t_2,t_3)$ and $\mathrm{cave}_{\mathscr{P}^\vee}(t_1,t_2,t_3)$ are both denormalized Lorentzian polynomials.

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